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Domain Decomposition and Iterative Refinement Methods for Mixed Finite Element Discretisations of Elliptic Problems

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Abstract

In this thesis, we first study the classical Schwarz alternating method and an additive, more parallel variant of it, known as the additive Schwarz method, applied to solve saddle point linear systems obtained by discretising a saddle point formulation of elliptic Neumann problems. We assume that the discretisation is obtained by using a mixed finite element method, in particular the Raviart-Thomas elements. We prove convergence with a rate independent of the mesh parameter h. We also present results of numerical experiments using these algorithms.

Following that, we study two algorithms to solve problems on iteratively refined meshes, namely the fast adaptive composite grid method (FAC), and the asynchronous fast adaptive composite grid method (AFAC). We give a proof of convergence of both these methods in the mixed finite element case, with a rate of convergence independent of the mesh parameters h_i. For the FAC algorithm, we also present quantitative bounds for the rate of convergence in the case of special discretisations, which show a fast rate of convergence, and one which is also independent of the geometry of the refined meshes, providing they are shape regular.

Finally, we study a Dirichlet-Neumann type algorithm for the mixed finite element case involving two non-overlapping subdomains. We use this as a preconditioner for a reduced Schur complement system obtained by using an algorithm of Glowinski and Wheeler. We prove that the rate of convergence is independent of h, the mesh parameter. We also present quantitative bounds in special geometries, that show a fast rate of convergence.

Contents

Chapter 1

A survey of background results, old and new.

Introduction.

In this thesis, we study certain iterative methods to solve the saddle point linear systems resulting from mixed finite element discretisations of second order linear elliptic Neumann problems. Unlike standard finite element discretisations of elliptic problems, which lead to large, symmetric, positive definite linear systems, the discretisation of saddle point formulations of elliptic problems, lead to large symmetric indefinite linear systems. The iterative methods we study are known as domain decomposition methods, and are based upon a division of the domain of the elliptic problem into smaller pieces, so called substructures, with or without overlap. The domain decomposition algorithms involve the solution of problems on each of the substructures, often concurrently, during each iteration. These methods are therefore well suited for parallel computers. We introduce several methods, and study their convergence in the mixed finite element case.

In this Chapter, we present the relevent background about saddle point formulations of elliptic problems, and mixed finite element methods using the Raviart-Thomas spaces. We then discuss an extension theorem for the Raviart-Thomas finite element spaces, which we use later in establishing bounds for the rate of convergence of the domain decomposition methods. A section containing some background on iterative methods, is also included.

In Chapter 2, we discuss algorithms involving subdomains with overlap, such as the classical Schwarz alternating method, cf. Lions [22], and the additive Schwarz method, as studied by Dryja and Widlund [13]. We present proofs of convergence

of these iterative methods when applied to the mixed finite element case, and also show that the rate of convergence is independent of the mesh parmeter h. We also present numerical results of tests using these methods with many subdomains and a certain coarse mesh model problem, which improves the rate of convergence. The results indicate a rate of convergence, which is independent of the mesh parameter h and even the number of subdomains. We have also tested the methods on problems in which the discontinuity in the coefficients of the elliptic operator is large. Such large variations in the coefficients occur in certain applications involving flow in porous media. The rate of convergence remains independent of the jump in the discontinuity, but the accuracy of the pressure deteriorates. See the section on numerical results, in Chapter 2.

Following that, in Chapter 3, we discuss certain iterative methods, known as the fast adaptive composite grid method (FAC) and the asynchronous fast adaptive composite grid method (AFAC), to solve problems on repeatedly refined meshes. These algorithms were originally developed for standard finite element discretisations of elliptic problems, cf. McCormick and Thomas [26], Mandel and McCormick [24] and Dryja and Widlund [14], [36]. We study these algorithms for mixed finite element methods. We show that the rate of convergence of the FAC algorithm is independent of the mesh parameters h_i, and also present some quantitative results showing that the rate of convergence is fast for certain discretisations. We also study a variant of the AFAC algorithm, and show that it has a rate of convergence independent of the mesh parameters h_i.

Finally, in Chapter 4, we study a domain decomposition method involving two subdomains without overlap, in the mixed finite element case. This is based on an algorithm of Glowinski and Wheeler [16], and ideas about a *Neumann-Dirichlet* preconditioner used in standard finite element discretisations of elliptic problems, cf. Bjørstad and Widlund [4] and Bramble, Pasciak and Schatz [5]. See also Quarteroni [29] for related work on saddle point problems. We prove that the *Dirichlet-Neumann* algorithm has a rate of convergence independent of the mesh parameter h, and also indicate that for certain geometries a fast rate of convergence is obtained.

1.1 Function spaces and trace theorems.

In this section, we introduce some function spaces and also present trace and extension theorems which will be used later on. All of the spaces introduced are Hilbert spaces. First we introduce the Sobolev spaces and some trace and extension theorems; cf. Lions and Magenes [21], Nečas [27], and Grisvard [19]. Then we introduce the $\vec{H}(div,.)$ space, some of its subspaces, and related trace and extension theorems. See Girault and Raviart [15], and Raviart and Thomas [30].

1.1.1 Sobolev spaces.

Let $\Omega \subset R^2$ be a Lipschitz region, i.e., $\partial\Omega$ can be represented locally as a Lipschitz function. We denote by

$$H^0(\Omega) \equiv L^2(\Omega) \equiv \{u : \int_\Omega |u|^2 dx < \infty\},$$

and define the norm by $||u||_{L^2(\Omega)} \equiv (\int_\Omega u^2 dx)^{1/2}$. For integer $m \geq 0$, we define the Sobolev space by,

$$H^m(\Omega) \equiv \{u \in L^2(\Omega) : \partial^\alpha u \in L^2(\Omega), \text{ for } |\alpha| \leq m\},$$

where $\alpha = (\alpha_1, \alpha_2)$ are non-negative integer indices representing the order of the partial derivitives; $|\alpha| \equiv \alpha_1 + \alpha_2$; and $\partial^\alpha u \equiv (\frac{\partial}{\partial x_1})^{\alpha_1}(\frac{\partial}{\partial x_2})^{\alpha_2} u$. The norm and semi-norm are defined respectively by

$$||u||_{H^m(\Omega)} \equiv (\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha u|^2 dx)^{1/2}, \quad |u|_{H^m(\Omega)} \equiv (\sum_{|\alpha|=m} \int_\Omega |\partial^\alpha u|^2 dx)^{1/2}.$$

For non integer s, with $m < s < m+1$, where m is a non-negative integer, we can define the fractional order Sobolev space $H^s(\Omega)$, and its associated norm by using the theory of interpolation spaces,

$$H^s(\Omega) \equiv [H^m(\Omega), H^{m+1}(\Omega)]_{(s-m)}.$$

These spaces form a Hilbert scale, and satisfy various interpolation inequalities, see Lions and Magenes [21].

For $0 < \epsilon < 1$, we also have the following equivalent definition of the fractional order Sobolev space $H^\epsilon(\Omega)$:

$$H^\epsilon(\Omega) \equiv \{u \in L^2(\Omega) : |u|_{H^\epsilon(\Omega)} < \infty\},$$

equipped with an equivalent semi-norm and norm defined respectively by

$$|u|_{H^\epsilon(\Omega)} \equiv (\int_{x\in\Omega}\int_{y\in\Omega}(|u(x)-u(y)|^2/|x-y|^{2+2\epsilon})dxdy)^{1/2},$$
$$\|u\|_{H^\epsilon(\Omega)} \equiv (\|u\|^2_{L^2(\Omega)} + |u|^2_{H^\epsilon(\Omega)})^{1/2}.$$

For $0 < s = m + \epsilon$, where $m \geq 0$ is an integer and $0 < \epsilon < 1$, we define the fractional order Sobolev space $H^s(\Omega)$ by,

$$H^s(\Omega) \equiv \{u \in H^m(\Omega) : \partial^\alpha u \in H^\epsilon(\Omega), \quad |\alpha| = m\},$$

and define an equivalent norm by

$$\|u\|_{H^s(\Omega)} \equiv (\|u\|^2_{H^m(\Omega)} + \sum_{|\alpha|=m} |\partial^\alpha u|^2_{H^\epsilon(\Omega)})^{1/2}.$$

Having defined the Sobolev spaces $H^s(\Omega)$ for non-negative s, we now present the definition of their dual spaces. For $s > 0$, we define the dual space $(H^s(\Omega))'$ as the space of continuous linear functionals on $H^s(\Omega)$:

$$(H^s(\Omega))' \equiv \{v(.) : H^s(\Omega) \longrightarrow R; \|v(.)\| < \infty\},$$

with the norm defined by

$$\|v\|_{(H^s(\Omega))'} \equiv \sup_{u\in H^s(\Omega)} \frac{|\int_\Omega uvdx|}{\|u\|_{H^s(\Omega)}}$$

where $\int_\Omega uvdx$ represents the duality pairing. These spaces form a Hilbert scale with a dense, compact imbedding:

$$H^{s_2}(\Omega) \hookrightarrow H^{s_1}(\Omega)$$

for $s_1 < s_2$.

Next, we discuss the Sobolev spaces of functions on the boundary $\partial\Omega$. Since, we will need to use these boundary spaces only for $-1 < s < 1$, we present their definition only for these values. $L^2(\partial\Omega)$ is defined using the standard definition of the boundary integral:

$$L^2(\partial\Omega) \equiv \{u : \int_{\partial\Omega} |u|^2 ds_x < \infty\},$$

with the norm defined by

$$\|u\|_{L^2(\partial\Omega)} \equiv (\int_{\partial\Omega} u^2 ds_x)^{1/2}.$$

For $0 < s < 1$ the Sobolev spaces on the boundary is defined by

$$H^s(\partial\Omega) \equiv \{u \in L^2(\partial\Omega) : |u|_{H^s(\partial\Omega)} < \infty\}, \tag{1.1}$$

where the semi-norm and norm are defined respectively by

$$\begin{aligned} |u|_{H^s(\partial\Omega)} &\equiv (\textstyle\int_{x\in\partial\Omega}\int_{y\in\partial\Omega}(|u(x)-u(y)|^2/|x-y|^{1+2s})ds_x ds_y)^{1/2}, \\ \|u\|_{H^s(\partial\Omega)} &\equiv (\|u\|^2_{L^2(\partial\Omega)} + |u|^2_{H^s(\partial\Omega)})^{1/2}. \end{aligned} \tag{1.2}$$

For $-1/2 \le s < 0$, we define the Sobolev space $H^s(\partial\Omega)$ by duality.

Let $\Gamma \subset \partial\Omega$ be a nonempty proper subset of $\partial\Omega$. For $s = \frac{1}{2}$, we introduce the interpolation space

$$H^{1/2}_{00}(\Gamma) \equiv [L^2(\Gamma), H^1_0(\Gamma)]_{1/2}.$$

The space $H^{1/2}_{00}(\Gamma) \neq H^{1/2}(\Gamma)$, but is continuously imbedded in $H^{1/2}(\Gamma)$:

$$H^{1/2}_{00}(\Gamma) \hookrightarrow H^{1/2}(\Gamma),$$

and has the property that functions in $H^{1/2}_{00}(\Gamma)$ can be extended by zero to the rest of $\partial\Omega$, as a continuous map, to a function in $H^{1/2}(\partial\Omega)$. An equivalent definition of the $H^{1/2}_{00}(\Gamma)$ norm is obtained by:

$$\|u\|^2_{H^{1/2}_{00}(\Gamma)} \equiv |u|^2_{H^{1/2}(\Gamma)} + \int_\Gamma (\frac{u^2(t)}{t} + \frac{u^2(t)}{L-t})ds,$$

where t is an arclength parameter along Γ and L is the length of Γ. One can verify using this formula and the definition of the $H^{1/2}(\partial\Omega)$ norm given by equation (1.1), that a function in $H^{1/2}(\partial\Omega)$ which is zero on $\partial\Omega - \Gamma$, has its $H^{1/2}(\partial\Omega)$ norm bounded by the $H^{1/2}_{00}(\Gamma)$ norm. Using this, we also see that functions in $H^{1/2}(\Gamma)$, when extended by zero to the rest of $\partial\Omega$, do not neccessarily belong to $H^{1/2}(\partial\Omega)$. The dual space will be denoted by $(H^{1/2}_{00}(\Gamma))'$. See Lions and Magenes [21], for details about the $H^{1/2}_{00}(.)$ spaces.

Next, we define the trace map, and state a trace theorem. For smooth functions defined on $\overline{\Omega}$, we have an obvious pointwise definition of its *trace* or *boundary values* on $\partial\Omega$. The following Lemma extends this notion of *trace* to $H^s(\Omega)$ functions. Since $\Omega \subset R^2$ is a Lipschitz region, we state the result only for $1/2 < s < 3/2$, however we remark that for smooth domains, the result is true for all $s > 1/2$. For a proof, see Nečas [27], or Grisvard [19].

Lemma 1 (Trace) *Let $1/2 < s < 3/2$. Then the trace map γ, with $\gamma u = u\mid_{\partial\Omega}$ for smooth functions u, can be extended as a continuous map from $H^s(\Omega)$ onto $H^{s-1/2}(\partial\Omega)$*

$$\gamma : H^s(\Omega) \xrightarrow{onto} H^{s-1/2}(\partial\Omega),$$

with a positive constant $c(\Omega, s)$ such that

$$\|\gamma u\|_{H^{s-1/2}(\partial\Omega)} \leq c(\Omega, s)\|u\|_{H^s(\Omega)}.$$

The constant $c(\Omega, s)$ depends upon s, and the Lipschitz constant of the region Ω.

Since γ maps $H^s(\Omega)$ continuously onto $H^{s-1/2}(\partial\Omega)$, we can apply the open mapping theorem to get a continuous right inverse E

$$E : H^{s-1/2}(\partial\Omega) \longrightarrow H^s(\Omega), \tag{1.3}$$

with

$$\gamma E g = g, \quad \forall g \in H^{s-1/2}(\partial\Omega).$$

This is an *extension* theorem.

Using the trace theorem. we are able to define a closed subspace of $H^1(\Omega)$. We let $H_0^1(\Omega)$ denote

$$H_0^1(\Omega) \equiv \{u \in H^1(\Omega) : \gamma u = 0 \text{ on } \partial\Omega\},$$

and it is provided with the $H^1(\Omega)$ norm. We now state Freidrichs' inequality about functions in $H_0^1(\Omega)$:

Lemma 2 (Freidrichs' inequality) *There exists a positive constant $c(\Omega)$ such that for all $u \in H_0^1(\Omega)$:*

$$\|u\|_{H^1(\Omega)} \leq c(\Omega)|u|_{H^1(\Omega)}.$$

We also present Poincaré's inequality.

Lemma 3 (Poincaré's inequality) *There exists a positive constant $c(\Omega, s)$ such that for all $u \in H^s(\Omega)$, for $1 \geq s > 0$, we have*

$$\|u\|_{H^s(\Omega)} \leq c(\Omega, s)[|u|^2_{H^s(\Omega)} + (\int_\Omega u dx)^2]^{1/2}.$$

Let $\mathcal{P}_r(\Omega)$ denote the polynomials of degree less than or equal to r on Ω. Then, by using Poincaré's inequality, it can be shown that the H^s-seminorm is equivalent to the H^s-norm on the quotient space $H^s(\Omega)/\mathcal{P}_0(\Omega)$. For a proof of this result, see Nečas [27].

We end this section, by introducing the Sobolev spaces $(H^s(\Omega))^2$ of vector functions, $\vec{u}$. Let $\vec{u} = (u_1, u_2)$. We associate the following norm on $(H^s(\Omega))^2$ for $s \geq 0$:

$$\|\vec{u}\|_{(H^s(\Omega))^2} \equiv \left(\|u_1\|^2_{H^s(\Omega)} + \|u_2\|^2_{H^s(\Omega)}\right)^{1/2}.$$

The dual spaces are defined analogously to the scalar case.

1.1.2 The $\vec{H}(div, \Omega)$ space and its subspaces.

We now introduce a function space that will be used in the mixed formulation of elliptic problems. Let $\vec{H}(div, \Omega)$ denote

$$\vec{H}(div, \Omega) \equiv \{\vec{u} \in (L^2(\Omega))^2 : \nabla \cdot \vec{u} \in L^2(\Omega)\},$$

equipped with the following norm

$$\|\vec{u}\|_{\vec{H}(div,\Omega)} \equiv \left(\|\vec{u}\|^2_{(L^2(\Omega))^2} + \|\nabla \cdot \vec{u}\|^2_{L^2(\Omega)}\right)^{1/2}.$$

Clearly, $(H^1(\Omega))^2$ is continuously imbedded in $\vec{H}(div, \Omega)$.

We now discuss the *normal component* of a vector function on the boundary of its domain of definition. For a Lipschitz domain Ω, the unit normal $\vec{n}$ to the boundary $\partial\Omega$ exists almost everywhere. Thus, for a smooth vector function $\vec{u}$ on $\overline{\Omega}$, the normal component $\vec{u} \cdot \vec{n}$ on $\partial\Omega$ exists almost everywhere. The following *normal trace* Lemma extends the notion of the *normal component* to $\vec{H}(div, \Omega)$ functions. We include a proof. Also, see Raviart and Thomas [30].

Lemma 4 *Given $\vec{u} \in \vec{H}(div, \Omega)$, there exists a map γ_n,*

$$\gamma_n : \vec{H}(div, \Omega) \xrightarrow{onto} H^{-1/2}(\partial\Omega),$$

such that for smooth vector functions $\vec{u}$, we have $\gamma_n \vec{u} = \vec{u} \cdot \vec{n}$ on $\partial\Omega$. There exists a constant $c(\Omega)$, which depends on the Lipschitz constant of the domain Ω, such that,

$$\|\gamma_n \vec{u}\|_{H^{-1/2}(\partial\Omega)} \leq c(\Omega)\|\vec{u}\|_{\vec{H}(div,\Omega)}, \quad \forall \vec{u} \in \vec{H}(div, \Omega).$$

PROOF OF THEOREM. Let $g \in H^{1/2}(\partial\Omega)$. Let $\phi = Eg$, be its extension into $H^1(\Omega)$, as defined by equation (1.3). Then ,

$$\nabla \cdot (\phi\vec{u}) = \phi(\nabla \cdot \vec{u}) + \vec{\nabla}\phi \cdot \vec{u}.$$

Integrating and using the divergence theorem, we obtain

$$\int_{\partial\Omega} g\gamma_n\vec{u}ds_x = \int_{\Omega}(\phi\nabla \cdot \vec{u} + \vec{\nabla}\phi \cdot \vec{u})dx,$$

since $\vec{n} \cdot \vec{u} = \gamma_n\vec{u}$. Applying the Cauchy-Schwarz inequality, we obtain

$$\left|\int_{\partial\Omega} g\gamma_n\vec{u}ds_x\right| \leq \|\phi\|_{H^1(\Omega)}\|\vec{u}\|_{\vec{H}(div,\Omega)},$$

and using the the boundedness of the extension map E, we obtain

$$\left|\int_{\partial\Omega} g\gamma_n\vec{u}ds_x\right| \leq c(\Omega)\|g\|_{H^{1/2}(\partial\Omega)}\|\vec{u}\|_{\vec{H}(div,\Omega)}.$$

Using the definition of the $H^{-1/2}(\partial\Omega)$ norm, we obtain

$$\|\gamma_n\vec{u}\|_{H^{-1/2}(\partial\Omega)} \leq c(\Omega)\|\vec{u}\|_{\vec{H}(div,\Omega)}.$$

This proves that γ_n is continuous.

We now show that γ_n maps $\vec{H}(div,\Omega)$ onto $H^{-1/2}(\partial\Omega)$. To do this, we pose the following Neumann problem for ϕ with Neumann data $g \in H^{-1/2}(\partial\Omega)$, using its variational formulation on $H^1(\Omega)$:

$$\begin{cases} -\Delta\phi + \phi &= 0 \quad \text{in } \Omega, \\ \partial\phi/\partial n &= g \quad \text{in } \partial\Omega, \end{cases}$$

By construction, $\phi \in H^1(\Omega)$. Also, from the equation we have $\vec{\nabla}\phi \in \vec{H}(div,\Omega)$ with $\gamma_n\vec{\nabla}\phi = g$. Thus γ_n maps $\vec{H}(div,\Omega)$ onto $H^{-1/2}(\partial\Omega)$. □

By an application of the open mapping theorem, we have a continuous right inverse E_n, extending $H^{-1/2}(\partial\Omega)$ functions into $\vec{H}(div,\Omega)$.

Next, we define the *divergence free* subspaces of the $\vec{H}(div,\Omega)$ spaces:

$$\vec{H}(div^0,\Omega) \equiv \{\vec{u} \in \vec{H}(div,\Omega) : \nabla \cdot \vec{u} = 0\}.$$

$\vec{H}(div^0,\Omega)$ is equipped with the $\vec{H}(div,\Omega)$ norm. We note that the $\vec{H}(div,\Omega)$ norm is equal to the $(L^2(\Omega))^2$ norm on $\vec{H}(div^0,\Omega)$.

Next, we define subspaces of $\vec{H}(div, \Omega)$ with zero *normal trace* on $\partial\Omega$:

$$\vec{H}_{0,\partial\Omega}(div, \Omega) \equiv \{\vec{u} \in \vec{H}(div, \Omega) : \gamma_n \vec{u} = 0\},$$

equipped with the $\vec{H}(div, \Omega)$ norm. Similarly, we denote by $\vec{H}_{0,\partial\Omega}(div^0, \Omega)$:

$$\vec{H}_{0,\partial\Omega}(div^0, \Omega) \equiv \{\vec{u} \in \vec{H}(div^0, \Omega) : \gamma_n \vec{u} = 0\},$$

equipped with the $\vec{H}(div, \Omega)$ or $(L^2(\Omega))^2$ norm.

Finally, we state a result about composite functions. See Raviart and Thomas [30]. Let $\overline{\Omega} = \cup_{j=1}^{N} \overline{\Omega_j}$ where the $\{\Omega_j\}$ are mutually disjoint. Then we have the following neccessary and sufficient condition for $\vec{v}$ to be in $\vec{H}(div, \Omega)$:

Lemma 5 *Suppose that $\vec{v}|_{\Omega_i} \in \vec{H}(div, \Omega_i)$ for $i = 1, \ldots, N$; and suppose that*

$$\begin{cases} \vec{n}_{ij} \cdot \vec{v}|_{\Omega_i} + \vec{n}_{ji} \cdot \vec{v}|_{\Omega_j} = 0 \ \ on\ \partial\Omega_i \cap \partial\Omega_j \quad \forall i, j, \\ where\ \vec{n}_{ij}\ is\ the\ outward\ normal\ from\ \Omega_i\ to\ \Omega_j\ on\ \partial\Omega_i \cap \partial\Omega_j. \end{cases} \tag{1.4}$$

Then $\vec{v} \in \vec{H}(div, \Omega)$.

PROOF OF LEMMA. Let $\vec{v}$ satisfy the above hypothesis. We will show that $\vec{v} \in \vec{H}(div, \Omega)$. First, since

$$\vec{v}|_{\Omega_i} \in (L^2(\Omega_i))^2, \forall i \Longrightarrow \vec{v} \in (L^2(\Omega))^2.$$

Next, we will show that the divergence of $\vec{v}$, in the sense of distributions, is in $L^2(\Omega)$, thereby making $\vec{v} \in \vec{H}(div, \Omega)$. Our candidate for $\nabla \cdot \vec{v}$, in the sense of distributions, is, simply the $L^2(\Omega)$ function $d(x)$ obtained by letting:

$$d(x)|_{\Omega_i} \equiv \nabla \cdot \vec{v}(x)|_{\Omega_i}, \quad \in L^2(\Omega_i),$$

for each Ω_i. Let $\phi \in C_0^\infty(\Omega)$ be an infinitely differentiable test function with compact support in Ω. Then, by the definition of the distributional derivitive, we have

$$\int_\Omega \phi \nabla \cdot \vec{v} dx \equiv -\int_\Omega \vec{\nabla}\phi \cdot \vec{v} dx.$$

Thus, we need to show that

$$\int_\Omega \phi \nabla \cdot \vec{v} dx - \int_\Omega \phi d(x) dx = 0.$$

Using the definition of $d(x)|_{\Omega_i}$ and of the distributional derivitive and integrating by parts, we obtain,

$$\int_\Omega \phi\nabla\cdot\vec{v}dx \equiv -\int_\Omega \vec{\nabla\phi}\cdot\vec{v}dx = -\int_\Omega \vec{\nabla\phi}\cdot\vec{v}dx - \sum_{i=1}^{N}\int_{\Omega_i}\phi\nabla\cdot\vec{v}dx,$$
$$= -\int_\Omega \vec{\nabla\phi}\cdot\vec{v}dx + \sum_{i=1}^{N}\int_{\Omega_i}\vec{\nabla\phi}\cdot\vec{v}dx - \sum_{i=1}^{N}\int_{\partial\Omega_i}\phi\vec{n}\cdot\vec{v}dx$$
$$= -\sum_{i=1}^{N}\int_{\partial\Omega_i}\phi\vec{n}\cdot\vec{v}dx.$$

Now by assumption, we have a cancellation of the sum of *normal traces* of $\vec{v}$ on $\partial\Omega_i\cap\partial\Omega_j$. Thus, this last term is zero. This proves that $\nabla\cdot\vec{v} = d(x)$, and therefore, $\nabla\cdot\vec{v}\in L^2(\Omega)$, since $d(x)\in L^2(\Omega)$. Thus, $\vec{v}\in\vec{H}(div,\Omega)$.

The neccessary part of the theorem is easily proved by using the *normal trace* Lemma , and the property that the restriction of a $\vec{H}(div,\Omega)$ function to a subdomain Ω_i is in $\vec{H}(div,\Omega_i)$. □

1.2 A saddle-point formulation of elliptic problems.

In this section, we introduce the elliptic problem that we consider throughout this thesis. First we describe the elliptic partial differential equation. We also present an abstract framework which is used to obtain the existence and uniqueness of solutions of the saddle point problem.

Then we describe its saddle-point variational formulation. Next, we describe how a discrete approximation to the saddle-point problem can be obtained by restricting the variational problem to finite dimensional subspaces. Finally, we describe the particular finite dimensional spaces we use, namely, the Raviart-Thomas finite element spaces.

Consider the following elliptic problem for p in a 2 dimensional polygonal region Ω:

$$\begin{cases} -\nabla\cdot(\mathcal{A}\nabla p) &= f \quad \text{in } \Omega \\ \vec{n}\cdot(\mathcal{A}\nabla p) &= g \quad \text{in } \partial\Omega, \end{cases} \tag{1.5}$$

where f and g satisfy the compatability condition $\int_\Omega f dx + \int_{\partial\Omega} g ds_x = 0$, and $\mathcal{A}(x)$ is a 2×2 symmetric matrix function with $L^\infty(\Omega)$ components satisfying

$$\lambda_{min}(\mathcal{A}(x)) \geq \alpha > 0,$$

almost everywhere, for some positive constant α.

Note that specifying the *conormal* derivitive of p, $\vec{n} \cdot \mathcal{A}\nabla p = g$, on $\partial\Omega$, is equal to specifying the *normal* derivitive, $\partial p/\partial n = g$ on $\partial\Omega$, when $\mathcal{A}(x) = I$, the identity. In applications to fluid flow in porous medium, where p denotes the pressure and $\mathcal{A}\nabla p$ denotes the fluid velocity given by Darcy's law, the *conormal* derivitive of p is the same as the *normal trace* of the velocity on $\partial\Omega$.

In such applications, approximations to the fluid velocity $\vec{u}$ can be obtained by discretising the elliptic problem in p and then $\mathcal{A}(x)\nabla p$. Alternatively, we can form an equivalent system using both p and $\mathcal{A}\nabla p$ as unknowns and study discretisations of the resulting system of partial differential equations. We present some details. Introduce $\vec{u} = \mathcal{A}\nabla p$ as an additional unknown. Then we obtain the transformed system:

$$\left\{ \begin{array}{rcll} \vec{u} & = & \mathcal{A}\nabla p & \text{in } \Omega \\ -\nabla \cdot \vec{u} & = & f & \text{in } \Omega \\ \vec{n} \cdot \vec{u} & = & g & \text{in } \partial\Omega \end{array} \right. \tag{1.6}$$

We may use finite difference formulas to discretise the enlarged system (1.6). However, in this thesis we consider only discretisations obtained by using the Raviart-Thomas finite elements, which in some cases are equivalent to finite difference approximations, see Wheeler and Gonzalez [33]. We introduce the variational problem associated with system (1.6). Without loss of generality, we assume that $g = 0$. Multiplying equation (1.6) by suitable test functions, and integrating by parts, we obtain the following equivalent variational form:

$$\left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } \vec{u} \in \vec{H}_{0,\partial\Omega}(div,\Omega),\ p \in L^2(\Omega)/R \text{ such that}} \\ \int_\Omega \vec{u}^T \mathcal{A}^{-1}(x)\vec{v}dx + \int_\Omega p(\nabla \cdot \vec{v})dx & = & 0, & \forall \vec{v} \in \vec{H}_{0,\partial\Omega}(div,\Omega) \\ \int_\Omega q(\nabla \cdot \vec{u})dx & = & -\int_\Omega fqdx, & \forall q \in L^2(\Omega)/R\ . \end{array} \right. \tag{1.7}$$

We now discuss an abstract framework for studying the saddle-point problem (1.7). See Girault and Raviart [15], for more details. Let X and Y be Hilbert spaces, let $a(.,.), b(.,.)$ be continuous bilinear forms, and let $W(.), F(.)$ be continuous linear functionals, defined on:

$$\begin{array}{rl} a: & X \times X \longrightarrow R \\ b: & X \times Y \longrightarrow R \\ W: & X \longrightarrow R \\ F: & Y \longrightarrow R. \end{array}$$

We then have the following existence and uniqueness theorem for the following prob-

lem:

$$\begin{cases} \text{Find } u \in X,\ p \in Y \text{ such that} & & \\ a(u,v) + b(v,p) & = W(v), & \forall v \in X \\ b(u,q) & = F(q), & \forall q \in Y\ , \end{cases} \tag{1.8}$$

Theorem 1 *Let a, b, W, F be defined as above. Suppose there exists a positive constant α such that*

$$a(u,u) \geq \alpha \|u\|_X^2, \quad \forall u \in X_0, \tag{1.9}$$

i.e., $a(.,.)$ is X_0-elliptic, where

$$X_0 \equiv \{v \in X : b(v,q) = 0, \quad \forall q \in Y\}. \tag{1.10}$$

Also suppose that there exists a positive constant β such that

$$\forall q \in Y, \quad \sup_{v \in X} \frac{b(v,q)}{\|v\|_X} \geq \beta \|q\|_Y, \tag{1.11}$$

i.e., $b(.,.)$ satisfies the Babuška-Brezzi inf-sup *condition. Then problem (1.8) is well posed, i.e., has a unique solution $(u,p) \in X \times Y$, satisfying the following bounds:*

$$\|u\|_X \leq c(\alpha,\beta)(\|F\|_{X'} + \|W\|_{Y'}),$$
$$\|p\|_Y \leq c(\alpha,\beta)(\|F\|_{X'} + \|W\|_{Y'}).$$

The constant $c(\alpha,\beta)$ depends only on α and β.

REMARK: By the continuity of the bilinear forms $a(.,.)$ and $b(.,.)$, there exists continuous linear maps

$$A : X \longrightarrow X'$$
$$B : X \longrightarrow Y',$$

defined by

$$\langle Au, v\rangle = a(u,v), \quad \forall u, v \in X,$$
$$\langle Bu, q\rangle = b(u,q), \quad \forall u \in X, q \in Y,$$

where $\langle .,. \rangle$ denotes the action of a linear functional.

REMARK: The *inf-sup* condition (1.11) is equivalent to the map B being an isomorphism between

$$B : X_0^{\perp} \longrightarrow Y',$$

with

$$\|Bv\|_{Y'} \geq \beta \|v\|_X, \quad \forall v \in X_0^{\perp}.$$

Thus if the *inf-sup* condition holds, then B maps X onto Y' and has a continuous right inverse, the norm of which can be estimated by $1/\beta$. Thus, given $q' \in Y'$, there exists $v \in X$ such that $Bv = q'$ satisfying

$$\|v\|_X \leq (1/\beta)\|q'\|_{Y'}.$$

REMARK: The *inf-sup* condition (1.11) is also equivalent to the map B^T being an isomorphism between

$$B^T : Y \longrightarrow (X_0^{\perp})',$$

with

$$\|B^T q\|_{X'} \geq \beta \|q\|_Y, \quad \forall q \in Y.$$

Thus, if the *inf-sup* condition holds, and $u' \in (X_0^{\perp})'$, i.e., $\langle u', v\rangle = 0$, for all $v \in X_0$, there exists $p \in Y$ with $B^T p = u'$ and satisfying

$$\|p\|_Y \leq (1/\beta)\|u'\|_{X'}.$$

To apply this abstract framework to the mixed formulation of the elliptic problem (1.7), to obtain well posedness, we need to check the X_0-ellipticity and the *inf-sup* condition for problem (1.7). In our application, we have

$$\begin{array}{rcl} X &=& \vec{H}_{0,\partial\Omega}(div, \Omega) \\ Y &=& L^2(\Omega)/R \\ a(\vec{u}, \vec{v}) &=& \int_\Omega \vec{v}(x)^T \mathcal{A}^{-1}(x)\vec{u}(x)dx \\ b(\vec{u}, p) &=& \int_\Omega (\nabla \cdot \vec{u}(x))p(x)dx \\ W(\vec{v}) &=& 0 \\ F(q) &=& -\int_\Omega f(x)q(x)dx, \end{array} \tag{1.12}$$

and,

$$X_0 = \vec{H}_{0,\partial\Omega}(div^0, \Omega).$$

First we check that $a(.,.)$ is X_0-elliptic. By assumption, $\lambda_{min}(\mathcal{A}(x)) \geq \alpha > 0$, for almost every x. Therefore, since for *divergence free* functions, the $\vec{H}(div,.)$ norm is equal to the $(L^2(.))^2$ norm, we have

$$a(\vec{u}, \vec{u}) \geq \alpha \|\vec{u}\|^2_{\vec{H}(div,\Omega)}, \quad \forall \vec{u} \in \vec{H}(div^0, \Omega). \tag{1.13}$$

Thus $a(.,.)$ is $\vec{H}_{0,\partial\Omega}(div^0, \Omega)$-elliptic.

Next, we check the *inf-sup* condition. A Lemma in Girault and Raviart [15], states that given $q \in L^2(\Omega)/R$, there exists $\vec{u} \in (H_0^1(\Omega))^2$ with $\nabla \cdot \vec{u} = q$ and satisfying

$$||\vec{u}||_{(H^1(\Omega))^2} \leq c||q||_{L^2(\Omega)},$$

for a positive constant c. By the imbedding of $(H_0^1(\Omega))^2$ in $\vec{H}(div, \Omega)$, we have $\vec{u} \in \vec{H}(div, \Omega)$ and we obtain an *inf-sup* condition for problem (1.7).

Thus the variational problem (1.7), the mixed formulation of an elliptic Neumann problem, has a unique solution in $\vec{H}_{0,\partial\Omega}(div, \Omega) \times L^2(\Omega)/R$. Note, since we are studying a Neumann boundary value problem, the pressure is unique only in $L^2(\Omega)/R$, i.e. unique up to a constant in $L^2(\Omega)$.

1.3 An abstract framework for the discretisation of the saddle-point problem.

Now that we have introduced the abstract variational problem (1.8), and the specific problem (1.7), that we study in this thesis, we discuss how to discretise the problem. For each h, let X_h and Y_h be finite dimensional subspaces of X and Y respectively. For each h we approximate the variational problem by restricting it to $X_h \times Y_h$ as follows:

$$\begin{cases} \text{Find } u_h \in X_h,\ p_h \in Y_h \text{ such that} & \\ a(u_h, v) + b(v, p_h) = W(v), & \forall v \in X_h \\ b(u_h, q) = F(q), & \forall q \in Y_h \end{cases} \tag{1.14}$$

Thus, for each h, (u_h, p_h) will be considered as an approximation to (u, p), the solution of problem (1.8). Here h denotes a parameter, usually the maximum mesh width, which tends to zero.

For the discrete variational problem (1.14) to be well posed, we assume that for each h there exists positive constants α_h, β_h, possibly dependent on h, such that the bilinear forms $a(.,.)$ and $b(.,.)$ satisfy the $X_{h,0}$-ellipticity and the Babuška-Brezzi *inf-sup* condition respectively:

$$\forall v_h \in X_{h,0}, \quad a(v_h, v_h) \geq \alpha_h ||v_h||_X^2,$$

$$\forall q \in Y_h, \quad \sup_{v \in X_h} \frac{b(v, q)}{||v||_X} \geq \beta_h ||q||_Y,$$

where

$$X_{h,0} \equiv \{v_h \in X_h : b(v_h, q) = 0, \forall q \in Y_h\}.$$

Then we have well posedness of problem (1.14) by Theorem 1. In addition, see Girault and Raviart [15], we have the following error bound:

$$\|u - u_h\|_X + \|p - p_h\|_Y \leq c(\alpha_h, \beta_h)\{\inf_{v_h \in X_h} \|u - v_h\|_X + \inf_{q_h \in Y_h} \|p - q_h\|_Y\} \tag{1.15}$$

where (u, p) are solutions of the continuous problem and (u_h, p_h) are solutions of the discrete problem. In most applications, if the spaces X_h and Y_h are chosen to approximate well X and Y respectively, then the approximation error can be shown to approach zero as h approaches zero.

This discretization leads to a saddle point problem, i.e., a symmetric indefinite linear system with the following block structure:

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} W_h \\ F_h \end{pmatrix}. \tag{1.16}$$

A_h is symmetric and positive definite and approximates the map A. It is uniformly positive definite, (in h), in the subspace $\{v_h : B_h v_h = 0\}$. B_h^T is the discrete approximation to the B map. Thus (u_h, p_h) can be found by solving the linear system (1.16).

We mention here that the spaces X_h and Y_h should be chosen so that the computation of the *stiffness* matrices (1.16) is tractable and further, that they approximate the original problem well. One such choice is the use of finite element spaces. Here we describe briefly their construction. Let us assume that the spaces X and Y in the abstract variational framework of Theorem 1 are function spaces defined on a polygonal domain Ω. For each h, where h is a real parameter, we partition the domain into triangles (quadrilaterals) with the width of each triangle (quadrilateral) being less than h. Then we define the spaces X_h and Y_h as consisting of piecewise smooth functions, usually polynomial in each triangle (quadrilateral), and satisfying some global regularity. In particular, throughout this thesis, for the mixed formulation of elliptic problems, (1.7), we use the Raviart-Thomas finite element spaces. For $r = 0, 1, 2, 3, \ldots$, these finite element spaces contain polynomials of order r, and satisfy a uniform *inf-sup* condition, i.e, these spaces satisfy the *inf-sup* condition with a positive constant β_h independent of h. We discuss these finite element spaces in more detail in the next section.

1.4 The Raviart-Thomas finite element spaces.

In this section, we introduce the Raviart-Thomas finite element spaces, see Raviart and Thomas [30], which we use to discretise the mixed formulation of elliptic problems (1.7). Throughout the rest of the thesis, we assume that Ω is a polygonal region in R^2. We partition Ω into disjoint subregions called elements, which are either all triangles or all parallelograms. Let h be a parameter denoting the maximum diameter of elements associated with the partition. We denote the partition or triangulation by τ^h, and the individual elements by K. The Raviart-Thomas finite element spaces, $V_h(\Omega)$ and $Q_h(\Omega)$, based on the triangulation τ^h, are subspaces of $\vec{H}(div,\Omega)$ and $L^2(\Omega)$, respectively, consisting of functions which are polynomial in each element K, and satisfy certain conditions. We describe them in more detail in the following subsections. We discuss two types of spaces, one based on triangular elements, and the other based on quadrilateral elements. We first define the spaces $V_h(\Omega)$ and $Q_h(\Omega)$, which are valid for Dirichlet boundary conditions. Then, we choose subspaces $X_h(\Omega)$ and $Y_h(\Omega)$ of $V_h(\Omega)$ and $Q_h(\Omega)$ respectively, which are valid for Neumann boundary value problems, the specific problem we consider throughout this thesis.

1.4.1 Triangular elements.

As before, we let Ω be a polygonal region in R^2. We partition it into triangles

$$\Omega = \cup_{K \in \tau^h} K,$$

where τ^h denotes the triangulation of Ω, as before. We assume that the vertices of the triangles either coincide with the vertices of other triangles or lie on the boundary $\partial\Omega$; In particular, we assume that the vertices do not lie on the edges of other triangles. For each triangle K, we denote by

$$\begin{aligned} h_K &\equiv \text{ Diameter of } K, \\ \rho_K &\equiv \text{ Diameter of circle inscribed in } K. \end{aligned}$$

DEFINITION. A family of triangulations $\{\tau^h\}$ of Ω is said to be *shape regular*, if there exists a constant $\sigma_1 > 0$, such that,

$$\forall h, \forall K \in \tau^h, \quad \sigma_1 \rho_K \geq h_K.$$

DEFINITION. A family of triangulations $\{\tau^h\}$ of Ω is said to be *uniformly shape regular*, if there exists positive constants σ_1, σ_2, such that:

$$\forall h, \forall K \in \tau^h, \quad \sigma_1 \rho_K \geq h_K \geq \sigma_2 h.$$

Affine maps and correspondences.

Let $\widehat{K}$ denote the unit triangle with vertices

$$\hat{a}_1 = (0,0); \ \hat{a}_2 = (1,0); \ \hat{a}_3 = (0,1).$$

Given a non-degenerate triangle $K \in \tau^h$ with vertices a_1, a_2, a_3, we map $\widehat{K}$ onto K, so that $F_K(\hat{a}_i) = a_i$, for $i = 1,2,3$, using an invertible affine linear map F_K:

$$F_K : \widehat{K} \longrightarrow K,$$

of the form $F_K(\hat{x}) \equiv B_K\hat{x} + b_K$, where B_K is a 2×2 invertible matrix and b_K is a 2-vector. Using the map $F_K(.)$, we can define a one-to-one *correspondence* between scalar and vector functions on $\widehat{K}$ and K.

For any scalar function $\hat{\phi}$ defined on $\widehat{K}$, we associate the scalar function ϕ defined on K by a one-to-one *correspondence* $\hat{\phi} \longleftrightarrow \phi$, given by:

$$\phi \equiv \hat{\phi} \circ F_K^{-1}, \quad (\hat{\phi} \equiv \phi \circ F_K).$$

For vector functions $\hat{\vec{v}} = (\hat{v}_1, \hat{v}_2)$, defined on $\widehat{K}$, however, to preserve the *normal trace* of certain functions, we define $\vec{v}$ on K by a one-to-one *correspondence* $\hat{\vec{v}} \longleftrightarrow \vec{v}$ given by:

$$\vec{v} \equiv (1/J_K)B_K\hat{\vec{v}} \circ F_K^{-1}, \quad (\hat{\vec{v}} \equiv J_K B_K^{-1}\vec{v} \circ F_K),$$

where $J_K \equiv det(B_K)$.

The following Lemma, found in Raviart and Thomas [30], states some of the properties of the *divergence* and the *normal trace* of vector functions under the *correspondences*.

Lemma 6 *For any function $\hat{\vec{v}} \in \vec{H}(div, \widehat{K})$, we have:*

$$\forall \hat{\phi} \in L^2(\widehat{K}), \qquad \int_{\hat{K}} \hat{\phi}\widehat{\nabla} \cdot \hat{\vec{v}} d\hat{x} = \int_K \phi \nabla \cdot \vec{v} dx,$$

$$\forall \hat{\phi} \in L^2(\partial\widehat{K}), \qquad \int_{\partial\hat{K}} \hat{\phi}\hat{\vec{n}} \cdot \hat{\vec{v}} ds_{\hat{x}} = \int_{\partial K} \phi\vec{n} \cdot \vec{v} ds_x.$$

Next, we state another Lemma found in Raviart and Thomas [30], which relates the norms and semi-norms of scalar and vector functions and their *correspondences.*

Lemma 7 *For any integer $l \geq 0$, we have:*

$$\forall \hat{\phi} \in H^l(\widehat{K}), \qquad |\hat{\phi}|_{H^l(\hat{K})} \leq \|B_K\|^l |J_K|^{-1/2} |\phi|_{H^l(K)},$$

$$\forall \vec{\hat{v}} \in (H^l(\widehat{K}))^2, \qquad |\vec{\hat{v}}|_{(H^l(\hat{K}))^2} \leq \|B_K\|^l \|B_K^{-1}\| |J_K|^{1/2} |\vec{v}|_{(H^l(K))^2},$$

where $\|B_K\|$ and $\|B_K^{-1}\|$ denote the spectral norm of the matrices B_K and B_K^{-1} respectively.

Construction of the Raviart-Thomas finite element spaces using the spaces defined on the reference element $\widehat{K}$.

Let r be a non-negative integer, and let $\widehat{K}$ denote the reference unit triangle. We outline the construction of the Raviart-Thomas spaces $V_h(\Omega)$ and $Q_h(\Omega)$, of order r, on Ω.

1. Let $V_h(\widehat{K})$ and $Q_h(\widehat{K})$ be finite dimensional linear spaces defined on the unit reference triangle $\widehat{K}$.

2. Using the reference spaces $V_h(\widehat{K})$ and $Q_h(\widehat{K})$, we define the restriction of the Raviart-Thomas spaces $V_h(\Omega)$ and $Q_h(\Omega)$ to any non-degenerate triangle $K \in \tau^h$, which we denote by $V_h(K)$ and $Q_h(K)$, respectively. We define $V_h(K)$ and $Q_h(K)$ by using the one-to-one *correspondences* obtained by using the affine linear map between $\widehat{K}$ and K, as discussed in the previous subsubsection. i.e.,

$$V_h(K) \equiv \{\vec{v} \in \vec{H}(div, K) : \vec{v} \longleftrightarrow \vec{\hat{v}} \in V_h(\widehat{K})\},$$

$$Q_h(K) \equiv \{\phi \in L^2(K) : \phi \longleftrightarrow \hat{\phi} \in Q_h(\widehat{K})\}.$$

3. To complete our construction of the Raviart-Thomas spaces on Ω, we define the spaces by:

$$V_h(\Omega) \equiv \{\vec{v} \in \vec{H}(div, \Omega) : \vec{v}|_K \in V_h(K)\},$$

$$Q_h(\Omega) \equiv \{q \in L^2(\Omega) : q|_K \in Q_h(K)\}.$$

In the next subsubsection, we define the Raviart-Thomas spaces on the reference element $\widehat{K}$.

Raviart-Thomas spaces $V_h(\widehat{K})$ and $Q_h(\widehat{K})$ defined on the reference element $\widehat{K}$.

Here we define the Raviart-Thomas spaces $V_h(\widehat{K})$ and $Q_h(\widehat{K})$. We remark that our notation for the spaces differs from that used in Raviart and Thomas [30], with the spaces $V_h(.)$ and $Q_h(.)$ interchanged. Let r be a non-negative integer, $r = 0, 1, 2, 3, \ldots$. The space $V_h(\widehat{K})$ of order r, is a finite dimensional linear space satisfying:

$$\begin{array}{ll} (a1) & (\mathcal{P}_r(\widehat{K}))^2 \subset V_h(\widehat{K}); \\ (a2) & dimension(V_h(\widehat{K})) = (r+1)(r+3); \\ (a3) & \vec{v} \in V_h(\widehat{K}) \Longrightarrow \widehat{\nabla} \cdot \vec{v} \in \mathcal{P}_r(\widehat{K}); \\ (a4) & \vec{v} \in V_h(\widehat{K}) \Longrightarrow \vec{n} \cdot \vec{v} \in \widehat{\mathcal{S}}_r; \\ (a5) & V_h(\widehat{K}) \cap \vec{H}(div^0, \widehat{K}) \subset (\mathcal{P}_r(\widehat{K}))^2; \end{array}$$

where

- $\mathcal{P}_r(\widehat{K})$ denotes the polynomials of degree $\leq r$, on $\widehat{K}$;
- $\widehat{\nabla}$ denotes the *divergence* in $\widehat{K}$;
- $\vec{n}$ denotes the unit outer normal to $\partial\widehat{K}$;
- $\widehat{\mathcal{S}}_r \equiv \{s(\hat{x}) \in L^2(\partial\widehat{K}) : s(\hat{x})|_{\hat{e}} \in \mathcal{P}_r(\hat{e}),\ \text{for each edge } \hat{e} \subset \partial\widehat{K}\}$.

The following Lemma is found in Raviart and Thomas [30], where also examples of such spaces are given.

Lemma 8 *For a space $V_h(\widehat{K})$ satisfying conditions $(a2)$ to $(a5)$, a function $\vec{v} \in V_h(\widehat{K})$ is uniquely determined by:*

$(b1)$ *the values of $\vec{v} \cdot \vec{n}$ at $(r+1)$ distinct points on each edge $\hat{e}_i$ of $\partial\widehat{K}$;*

$(b2)$ *the moments of order $\leq r-1$ of $\vec{v}$, i.e.,*
$\int_{\widehat{K}} \vec{v} \cdot \vec{\wp}(\hat{x}) d\hat{x}, \quad \forall \vec{\wp}(\hat{x}) \in (\mathcal{P}_{r-1}(\widehat{K}))^2$.

We define the Raviart-Thomas pressure space $Q_h(\widehat{K})$ by:

$$Q_h(\widehat{K}) \equiv \mathcal{P}_r(\widehat{K}). \tag{1.17}$$

Note that, using the definition of the *correspondence* for scalar functions, we obtain that $Q_h(K) = \mathcal{P}_r(K)$.

REMARK. Condition $(b1)$ can equivalently be replaced by specifying the moments of order $\leq r$ on $\hat{e}_i$. i.e.,

$$\int_{\hat{e}_i} \hat{\phi}(\vec{v} \cdot \vec{n}) ds_{\hat{x}}, \quad \forall \hat{\phi} \in \mathcal{P}_r(\hat{e}_i),$$

for each edge $\hat{e}_i \subset \partial\widehat{K}$.

A basis for $V_h(\widehat{K})$.

Applying Lemma 8, we construct a basis for $V_h(\widehat{K})$ as follows. First, for $i = 1,2,3$ and $j = 0,\ldots,r$, let $\{\hat{\phi}_{i,j}\}$ denote an orthonormal piece-wise polynomial basis for $\mathcal{S}_r(\partial\widehat{K})$ equipped with the $L^2(\partial\widehat{K})$ inner product. On each edge $\hat{e}_i \subset \partial\widehat{K}$, let $\{\hat{\phi}_{i,j}\}_{j=0}^{j=r} \in \mathcal{P}_r(\hat{e}_i)$ be the Legendre polynomials of degree $\leq r$ on $\hat{e}_i$ extended by zero on the other edges, $\hat{e}_l, l \neq i$. This accounts for $3(r+1)$ basis elements. For the remaining $r(r+1)$ data, we use the nontrivial distinct moments

$$\int_{\hat{K}} \vec{v} \cdot \vec{\wp}_l d\hat{x},$$

of order $\leq r-1$ where $\{\vec{\wp}_l\}_{l=1}^{r(r+1)}$ form a basis for $(\mathcal{P}_{r-1}(\widehat{K}))^2$.

To construct the basis, for $i = 1,2,3$ and $j = 0,\ldots,r$, we let, $\{\vec{\hat{u}}_{i,j}\}$ denote $3(r+1)$ linearly independent functions in $V_h(\widehat{K})$ satisfying:

$$\begin{cases} \gamma_n \vec{\hat{v}}_{i,j} = \vec{\hat{n}} \cdot \vec{\hat{v}}_{i,j} = \hat{\phi}_{i,j} \text{ on } \partial\widehat{K}, \\ \int_{\hat{K}} \vec{\hat{v}}_{i,j} \cdot \vec{\wp}_l d\hat{x} = 0, \quad \forall l. \end{cases}$$

We pick the remaining $r(r+1)$ linearly independent basis elements $\{\vec{\hat{u}}_l\}_{l=1}^{r(r+1)}$ in $V_h(\widehat{K})$, as follows:

$$\begin{cases} \gamma_n \vec{\hat{u}}_l = 0, \quad \forall l, \\ \int_{\hat{K}} \vec{\hat{u}}_l \cdot \vec{\wp}_j d\hat{x} = \delta_{lj}, \quad \forall l, j = 1,\ldots,r(r+1). \end{cases}$$

Here, δ_{ij} denotes the Kronecker delta function,

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases}$$

Thus, we have constructed $3(r+1) + r(r+1)$ linearly independent functions which forms a basis for $V_h(\widehat{K})$. We will denote them by $\{\vec{\hat{u}}_{i,j}\} \cup \{\vec{\hat{u}}_l\}$, for $i = 1,2,3$, $j = 1,\ldots,(r+1)$, and $l = 1,\ldots,r(r+1)$.

A Dual Basis for $(V_h(\widehat{K}))'$.

Let $\{L_{i,j}(.)\} \cup \{L_l(.)\}$ denote an algebraic *dual basis* with respect to the basis $\{\vec{\hat{u}}_{i,j}\} \cup \{\vec{\hat{u}}_l\}$ for $V_h(\widehat{K})$, i.e., the $\{L_{i,j}(.)\} \cup \{L_l(.)\}$ denote linear functionals on $V_h(\widehat{K}) \subset C^\infty(\widehat{K})$, and form a basis for the dual space $V_h(\widehat{K})'$ satisfying:

$$\begin{aligned} L_{i,j}(\vec{\hat{u}}_{i',j'}) &= \delta_{i,i'}\delta_{j,j'}, & \forall i,j,i',j', \\ L_{i,j}(\vec{\hat{u}}_l) &= 0, & \forall i,j,l, \\ L_l(\vec{\hat{u}}_{i,j}) &= 0, & \forall i,j,l \\ L_l(\vec{\hat{u}}_{l'}) &= \delta_{l,l'}, & \forall l,l'. \end{aligned}$$

Thus we can express any element $\vec{\hat{v}} \in V_h(\widehat{K})$ by:

$$\vec{\hat{v}} = \sum_{i,j} L_{i,j}(\vec{\hat{v}})\vec{\hat{u}}_{i,j} + \sum_{l} L_l(\vec{\hat{v}})\vec{\hat{u}}_l.$$

REMARK. We can express the dual basis explicitly by,

$$L_{i,j}(\vec{\hat{v}}) = \int_{\hat{e}_i} \hat{\phi}_{i,j}\vec{\hat{n}} \cdot \vec{\hat{v}} ds_{\hat{x}},$$

and

$$L_l(\vec{\hat{v}}) = \int_{\hat{K}} \vec{\wp}_l \cdot \vec{\hat{v}} d\hat{x}.$$

An example of the lowest order Raviart-Thomas spaces defined on triangles.

We present the lowest order Raviart-Thomas spaces, i.e., for $r = 0$ on triangles. We have,

$$V_h(\widehat{K}) \equiv \{ \begin{pmatrix} a + b\hat{x}_1 \\ c + b\hat{x}_2 \end{pmatrix} : \forall a, b, c \in R; \text{ where } \hat{x} = (\hat{x}_1, \hat{x}_2) \in \widehat{K}\},$$

$$Q_h(\widehat{K}) \equiv \mathcal{P}_0(\widehat{K}).$$

For the velocity space $V_h(\Omega)$, we have

$$V_h(\Omega) = \{\vec{v} \in \vec{H}(div, \Omega) : \vec{v}|_K = (1/J_K)B_K\vec{\hat{v}}, \text{ for some } \vec{\hat{v}} \in V_h(\widehat{K})\}.$$

And for the lowest order pressure space $Q_h(\Omega)$, we have

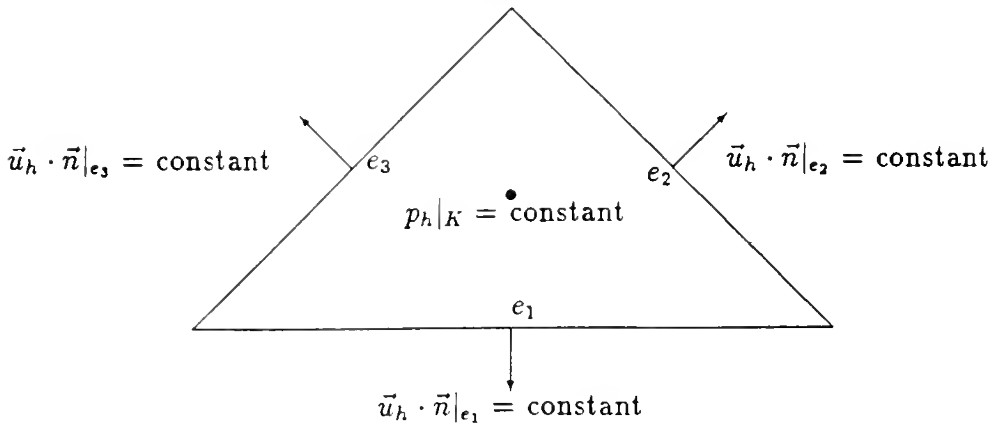
$$Q_h(\Omega) = \{q \in L^2(\Omega) : q|_K \in \mathcal{P}_0(K)\}.$$

Given the edges of the triangles forming the triangulation τ^h of Ω, the velocity $\vec{u}_h$ is uniquely specified by its normal components on the edges, which in this lowest order case is constant on the edges. See Figure 1.1. A basis for $Q_h(\Omega)$ is easily chosen; the ith basis function is one on the i^{th} triangle and zero on all other triangles.

The interpolation map.

We end this subsection by introducing an interpolation map associated with the Raviart-Thomas velocity space $V_h(\Omega)$. We will first define the interpolation map $\widehat{\Pi}$

Figure 1.1: Lowest order Raviart-Thomas spaces on a triangle.



from $(H^1(\widehat{K}))^2$ onto $V_h(\widehat{K})$. We discuss briefly why this is a continuous map. Then we define the interpolation map Π^h from $(H^1(\Omega))^2$ onto $V_h(\Omega)$, by letting

$$(\Pi^h\vec{v})|_K \longleftrightarrow \widehat{\Pi}\hat{\vec{v}}, \quad \forall K \in \tau^h. \tag{1.18}$$

We need to check that this is a valid definition, i.e., the defined function is in $\vec{H}(div, \Omega)$. This will follow by construction, from the compatible *normal trace* along inter-element boundaries, see Lemma 5. The interpolation map Π^h will also be shown to be continuous.

DEFINITION OF $\widehat{\Pi}$. By Lemma 8, given a vector function $\hat{\vec{v}} \in (H^1(\widehat{K}))^2$, we can determine a unique interpolant $\widehat{\Pi}\hat{\vec{v}} \in V_h(\widehat{K})$ satisfying:

$$\begin{array}{lll} (c1) & \forall\hat{\phi} \in \mathcal{P}_r(\hat{e}), & \int_{\hat{e}}(\widehat{\Pi}\hat{\vec{v}} - \hat{\vec{v}}) \cdot \hat{\vec{n}}\hat{\phi}ds_{\hat{x}} = 0, \quad \forall \text{ sides } \hat{e} \subset \partial\widehat{K}, \\ (c2) & \forall(\wp_1, \wp_2) \in (\mathcal{P}_{r-1}(\widehat{K}))^2, & \int_{\widehat{K}}(\widehat{\Pi}\hat{\vec{v}} - \hat{\vec{v}}) \cdot (\wp_1, \wp_2)d\hat{x} = 0. \end{array}$$

We show that these conditions are well defined for $(H^1(\widehat{K}))^2$ functions and that the interpolation map is continuous. We use the property that for functions $\hat{\vec{v}} \in (H^1(\widehat{K}))^2$, its *normal trace* $\gamma_n\hat{\vec{v}} \in H^{1/2-\epsilon}(\partial\widehat{K})$, for $\epsilon > 0$. This follows by:

- An application of the trace Lemma which gives $\gamma\hat{\vec{v}} \in (H^{1/2}(\partial\widehat{K}))^2$.
- Using

  $$\gamma_n\hat{\vec{v}} = \hat{\vec{n}} \cdot \gamma\hat{\vec{v}}.$$

  For the reference triangle $\widehat{K}$ the unit outer normal $\hat{\vec{n}}$ is constant on each edge $\hat{e}$. Thus for each edge $\hat{e}$, we have

  $$\gamma_n\hat{\vec{v}}|_{\hat{e}} \in H^{1/2}(\hat{e}).$$

- Using the definition of $H^{1/2-\epsilon}(\partial\widehat{K})$ on the one dimensional boundary $\partial\widehat{K}$ to get the result.

REMARK. Note that, a constant vector, which is a function in $(H^1(\widehat{K}))^2$, has a piecewise constant *normal trace*, since the *normal* is discontinuous across the vertices of the triangle $\widehat{K}$. Using the definition of the $H^{1/2}(\partial\widehat{K})$ norm, we see that piecewise constant functions on $\partial\widehat{K}$ are not in $H^{1/2}(\partial\widehat{K})$, due to unbounded norms.

REMARK. Note that, $\gamma_n\Pi^h\vec{u}|_{\partial K}$, by the definition of Π^h, is the affine transformation of a polynomial of degree less than or equal to r, on each edge $e \subset \partial K$. Also, by the definition of the interpolation map the inner product of the *normal trace* of the interpolant $\widehat{\Pi}\vec{v}$ with functions in $\hat{\mathcal{S}}_r$ equals the inner product of the same functions with the *normal trace* of $\hat{\vec{u}}$. This, together with Lemma 6 about *correspondences*, imply that on the boundary between adjacent elements the *normal trace* of the interpolant $\Pi^h\vec{u}$ is compatible. Thus, the interpolation map is well defined.

Now we note that we can also express the reference interpolation map in terms of the basis and *dual basis*:

$$\widehat{\Pi}\vec{v} = \sum_{i,j} L_{i,j}(\vec{v})\hat{\vec{u}}_{i,j} + \sum_{l} L_l(\vec{v})\hat{\vec{u}}_l.$$

By using the fact that the *normal trace* of $(H^1(\widehat{K}))^2$ functions are in $H^{1/2-\epsilon}(\partial\widehat{K})$, the linear functionals in the *dual basis* can be shown to be continuous on $(H^1(\widehat{K}))^2$, and thus, $\widehat{\Pi}$ is continuous on $(H^1(\widehat{K}))^2$. This, together with the *correspondences* between the various norms involved on K and $\widehat{K}$ imply the continuity of Π^h, cf. Raviart and Thomas [30].

Lemma 9 *Let $\vec{u} \in (H^1(\Omega))^2$, and let Π^h denote the interpolation operator defined by equation 1.18. Then, there exists a positive constant* c *such that:*

$$||\Pi^h\vec{u}||_{\vec{H}(div,\Omega)} \leq c||\vec{u}||_{(H^1(\Omega))^2}.$$

We present details of the proof later in this Chapter, when we prove the continuity of the interpolation map on $\vec{H}(div^0,\Omega) \cap (H^\epsilon(\Omega))^2$.

1.4.2 Quadrilateral elements.

In this section, we discuss the Raviart-Thomas spaces on triangulations consisting of quadrilateral elements. See Raviart and Thomas [30] and Thomas [32]. Let $\widehat{K}$ denote

the unit reference square with vertices,

$$\hat{a}_1 = (0,0);\ \hat{a}_2 = (1,0);\ \hat{a}_3 = (1,1);\ \hat{a}_4 = (0,1).$$

As in the previous section, the Raviart-Thomas spaces $V_h(\Omega)$ and $Q_h(\Omega)$ are constructed on Ω as $\vec{H}(div, \Omega)$ and $L^2(\Omega)$ subspaces, respectively, using the reference spaces $V_h(\widehat{K})$ and $Q_h(\widehat{K})$, and the invertible affine linear map $F_K = B_K\hat{x} + b_K$. Since affine linear functions map squares onto parallelograms, we assume that the elements K in the triangulation τ^h of Ω are parallelograms.

Let r be a non-negative integer. We present the definition of the reference spaces, $V_h(\widehat{K})$ and $Q_h(\widehat{K})$ of order r. First, for non-negative integers m, n, let

$$\mathcal{P}_{m,n}(\widehat{K}) \equiv \{\sum_{i=0}^{m}\sum_{j=0}^{n} c_{ij}\hat{x}_1^i\hat{x}_2^j : c_{ij} \in R;\ \text{where } \hat{x} = (\hat{x}_1, \hat{x}_2) \in \widehat{K}\}.$$

We let

$$V_h(\widehat{K}) \equiv \{(\hat{v}_1, \hat{v}_2) : \hat{v}_1 \in \mathcal{P}_{r+1,r}(\widehat{K}), \hat{v}_2 \in \mathcal{P}_{r,r+1}(\widehat{K})\},$$

and

$$Q_h(\widehat{K}) \equiv \mathcal{P}_{r,r}(\widehat{K}).$$

Note that

- $\vec{\hat{u}} \in V_h(\widehat{K}) \Longrightarrow \nabla \cdot \vec{\hat{u}} \in Q_h(\widehat{K}) = \mathcal{P}_{r,r}(\widehat{K})$.
- $\vec{\hat{u}} \in V_h(\widehat{K}) \Longrightarrow \gamma_n\vec{\hat{u}}|_{\hat{e}_i} \in \mathcal{P}_r(\hat{e}_i)$, for each edge $\hat{e}_i \subset \partial\widehat{K}$.

The following Lemma is stated in Raviart and Thomas [30].

Lemma 10 *A function $\vec{\hat{u}} \in V_h(\widehat{K})$ is uniquely determined by:*
(1) *the values of $\gamma_n\vec{\hat{u}}$ at $(r+1)$ distinct points on each side $\hat{e}_i \subset \widehat{K}$,*
(2) *the moments*

$$\int_{\hat{K}} \hat{u}_1\hat{x}_1^i\hat{x}_2^j d\hat{x},\quad 0 \le r \le r-1,\quad 0 \le j \le r,$$

$$\int_{\hat{K}} \hat{u}_2\hat{x}_1^i\hat{x}_2^j d\hat{x},\quad 0 \le r \le r,\quad 0 \le j \le r-1.$$

Condition (1) can equivalently be replaced by the moments of order $\le r$ on each edge $\hat{e}_i \subset \partial\widehat{K}$:

$$\int_{\hat{e}_i} \hat{\phi}\gamma_n\vec{\hat{u}}ds_{\hat{x}},\quad \forall\hat{\phi} \in \mathcal{P}_r(\hat{e}_i),\quad i = 1,2,3,4.$$

As for the case of triangular elements, we define

$$V_h(K) \equiv \{\vec{v} : \vec{v} \longleftrightarrow \hat{\vec{v}}, \text{ where } \hat{\vec{v}} \in V_h(\widehat{K})\},$$

and

$$Q_h(K) \equiv \{\phi : \phi \longleftrightarrow \hat{\phi}, \text{ where } \hat{\phi} \in Q_h(\widehat{K})\}.$$

Finally, we define

$$V_h(\Omega) \equiv \{\vec{v} \in \vec{H}(div, \Omega) : \vec{v}|_K \in V_h(K), \forall K \in \tau^h\},$$

and

$$Q_h(\Omega) \equiv \{\phi \in L^2(\Omega) : \phi|_K \in Q_h(K), \forall K \in \tau^h\}.$$

Most of the results holding for the triangular case holds also for the quadrilateral case.

We present an example of the lowest order Raviart-Thomas spaces based on a rectangular mesh.

$$V_h(\widehat{K}) \equiv \{\begin{pmatrix} a + b\hat{x}_1 \\ c + d\hat{x}_2 \end{pmatrix} : \forall a, b, c, d \in R; \text{ where } \hat{x} = (\hat{x}_1, \hat{x}_2) \in \widehat{K}\},$$

and

$$Q_h(\widehat{K}) \equiv \mathcal{P}_0(\widehat{K}).$$

As stated in Lemma 10, the degrees of freedom of a function in $V_h(\widehat{K})$ can be specified by the values of $\hat{\vec{v}} \cdot \hat{\vec{n}}$ at the midpoint on each of the four sides of $\widehat{K}$. Of course, $\hat{\vec{v}} \cdot \hat{\vec{n}}$ is constant on each edge, for the lowest order case. See Figure 1.2.

1.4.3 Convergence results and other properties.

We list some of the properties of the Raviart-Thomas spaces $V_h(\Omega)$ and $Q_h(\Omega)$, which we use later on.

The functions in $V_h(\Omega)$, which are *discrete divergence free*, are *divergence free* in the $L^2(\Omega)$ sense, i.e., if $\vec{v}_h \in V_h(\Omega)$ and

$$\int_\Omega \phi(\nabla \cdot \vec{v}_h) dx = 0, \quad \forall \phi \in Q_h(\Omega), \text{ then } \nabla \cdot \vec{v}_h = 0.$$

PROOF. From the definition of the reference spaces, we see that,

$$\hat{\vec{v}}_h \in V_h(\widehat{K}) \Longrightarrow \widehat{\nabla} \cdot \hat{\vec{v}} \in Q_h(\widehat{K}).$$

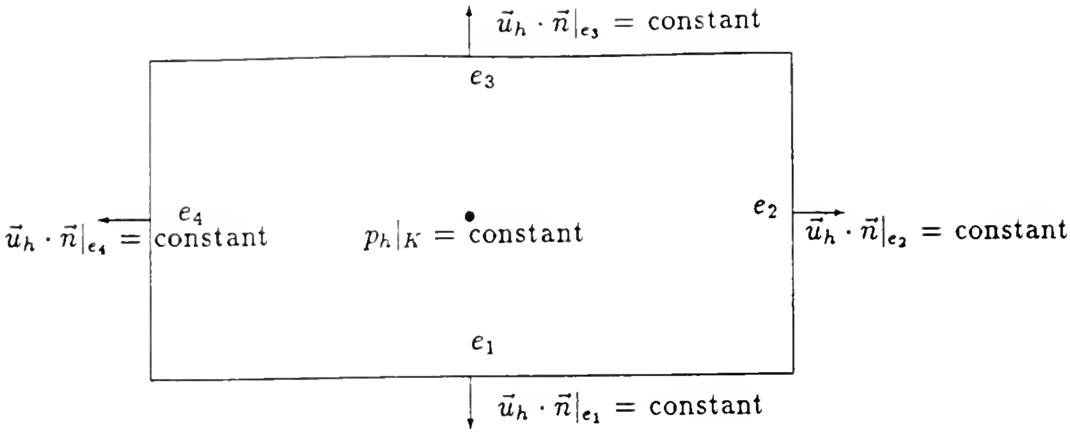


Figure 1.2: Lowest order Raviart-Thomas spaces on rectangles.

Thus,

$$\int_\Omega \phi(\nabla \cdot \vec{v}_h)dx = 0, \quad \forall \phi \in Q_h(\Omega)$$

$$\Longrightarrow \int_K \phi(\nabla \cdot \vec{v}_h)dx = 0, \quad \forall \phi \in Q_h(K)$$

$$\Longrightarrow \int_{\hat{K}} \hat{\phi}\widehat{\nabla} \cdot \vec{\hat{v}}_h d\hat{x} = 0, \quad \forall \hat{\phi} \in Q_h(\widehat{K}).$$

By substituting $\hat{\phi} = \widehat{\nabla} \cdot \vec{\hat{v}}_h$, we obtain $\widehat{\nabla} \cdot \vec{\hat{v}}_h = 0$.

$$\Longrightarrow (\nabla \cdot \vec{v}_h)|_K = 0, \quad \forall K \in \tau^h.$$

i.e., $(\nabla \cdot \vec{v}_h) = 0$. Hence, for the Raviart-Thomas spaces, a *discrete divergence free* function is *divergence free* in $L^2(\Omega)$. □

An additional property is:

If $\Pi^h \vec{v}$ is defined for a $\vec{H}(div^0, \Omega)$ function $\vec{v}$, then $\Pi^h \vec{v} \in \vec{H}(div^0, \Omega)$. i.e., the interpolation map, when defined, takes *divergence free* functions into *divergence free* functions.

PROOF. See Raviart and Thomas [30]. We show that when the interpolation map is well defined for a *divergence free* function $\vec{u}$ its interpolant $\Pi^h \vec{u}$ is also *divergence free*. Consider

$$\int_{\hat{K}} \hat{q} \nabla \cdot \hat{\Pi}\vec{\hat{v}} d\hat{x} = \int_{\partial \hat{K}} \vec{\hat{v}} \cdot \vec{\hat{n}} \hat{q} ds_{\hat{x}} - \int_{\hat{K}} \hat{\Pi}\vec{\hat{v}} \cdot \hat{\nabla}\hat{q} d\hat{x}. \tag{1.19}$$

For $\hat{q} \in \mathcal{P}_r(\widehat{K})$, for triangles, (or $\hat{q} \in \mathcal{P}_{r,r}(\widehat{K})$ for the quadrilateral case,) we obtain that $\hat{\nabla}\hat{q} \in (\mathcal{P}_{r-1}(\widehat{K}))^2$, (and $\hat{\nabla}\hat{q} \in \mathcal{P}_{r-1,r}(\widehat{K}) \times \mathcal{P}_{r,r-1}(\widehat{K})$, for the quadrilateral case.) For the same $\hat{q}$, we obtain that its restriction to any edge $\hat{e}$ in the boundary is a polynomial in $\mathcal{P}_r(\hat{e})$. Both these imply that we can replace $\hat{\Pi}\hat{u}$ by $\hat{u}$ in equation (1.19),

by using the definition of the reference interpolation map $\hat{\Pi}$. By an application of the *divergence* theorem, we obtain that this expression equals zero, since $\hat{\nabla} \cdot \hat{\vec{u}} = 0$. Thus, the interpolant is *discrete divergence free* which implies divergence free in the L^2 sense.

We now check whether the appropriate discrete subspaces for the Neumann boundary value problem (1.7) satisfies the *ellipticity* and *uniform inf-sup* conditions. Of course, this would be a sufficient condition for the convergence of the discrete problem. REMARK. For Neumann boundary conditions, the appropriate discrete subspaces of $\vec{H}_{0,\partial\Omega}(div, \Omega)$ and $L^2(\Omega)/R$ used in discretising problem (1.7) are

$$X_h(\Omega) \equiv V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div, \Omega), \text{ and } Y_h(\Omega) \equiv Q_h(\Omega) \cap (L^2(\Omega)/R),$$

respectively, where $V_h(\Omega)$ and $Q_h(\Omega)$ denote the Raviart-Thomas finite element spaces of order r based on the triangulation τ^h.

REMARK. Note that since $a(.,.)$ is $\vec{H}_{0,\partial\Omega}(div^0, \Omega)$-*elliptic* with a constant α, so is its subspace $X_h(\Omega)$, with the same constant α. See equation (1.13).

REMARK. A lemma in Raviart and Thomas [30] states that given $q_h \in Q_h$, there exists $\vec{v}_h \in V_h(\Omega)$ such that

$$\begin{aligned} \nabla \cdot \vec{v}_h &= q_h, \\ \|\vec{v}_h\|_{\vec{H}(div,\Omega)} &\leq c\|q_h\|_{L^2(\Omega)}, \end{aligned} \tag{1.20}$$

where $c > 0$ is a constant independent of h. This is equivalent to a *uniform inf-sup* condition for Dirichlet boundary value problems. However, for Neumann boundary value problems, given $q_h \in Q_h(\Omega) \cap L^2(\Omega)/R$, we need to find $\vec{v}_h$ satisfying the above bounds, but with $\vec{v}_h \in X_h(\Omega) = V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div, \Omega)$. See Thomas [32]. We give a proof using an extension theorem for the Raviart-Thomas velocity space $V_h(\Omega)$. See Theorem 2, discussed in the next section.

By equation (1.20), given $q_h \in Q_h(\Omega) \cap L^2(\Omega)/R$, there exists $\vec{v}_h \in V_h(\Omega)$ satisfying the above bounds. But $\gamma_n \vec{v}_h \neq 0$, in general. However, since $\nabla \cdot \vec{v}_h = q_h$, we have by an application of the divergence theorem that $\int_{\partial\Omega} \gamma_n \vec{v}_h ds_x = 0$. Now, by an extension theorem discussed in the next section, there exists $E^h \gamma_n \vec{v}_h \in V_h(\Omega) \cap \vec{H}(div^0, \Omega)$ such that $\nabla \cdot (E^h \gamma_n \vec{v}_h) = 0$ and such that

$$\|E^h \gamma_n \vec{v}_h\|_{\vec{H}(div,\Omega)} \leq c(\Omega)\|\gamma_n \vec{v}_h\|_{H^{-1/2}(\partial\Omega)}.$$

Here $c(\Omega)$ is a positive constant independent of h. By an application of the *normal tracc* lemma, we obtain

$$||E^h \gamma_n \vec{v}_h||_{\vec{H}(div,\Omega)} \leq c(\Omega)||\vec{v}_h||_{\vec{H}(div,\Omega)}.$$

Thus letting $\vec{u}_h = \vec{v}_h - E^h \gamma_n \vec{v}_h$, we obtain

$$\begin{aligned} \nabla \cdot \vec{u}_h &= q_h, \\ ||\vec{v}_h||_{\vec{H}(div,\Omega)} &\leq c||q_h||_{L^2(\Omega)}, \end{aligned}$$

where $c > 0$ is another constant independent of h. Note that, $\gamma_n \vec{u}_h = 0$. This shows that the *divergence* map

$$\nabla \cdot : V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div,\Omega) \xrightarrow{onto} Q_h(\Omega) \cap L^2(\Omega)/\mathcal{P}_0(\Omega),$$

is an isomorphism from $V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div,\Omega)$ onto $Q_h(\Omega) \cap L^2(\Omega)/\mathcal{P}_0(\Omega)$. This is equivalent to the *uniform inf-sup condition*, by one of the remarks following Theorem 1.

Approximation error for the Raviart-Thomas finite element spaces.

The following result is found in Raviart and Thomas [30].

Lemma 11 *Suppose that $p \in H^{r+2}(\Omega)$ and $\Delta p \in H^{r+1}(\Omega)$, for some integer $r \geq 0$. Then the discretisation error using the Raviart-Thomas finite element spaces of order r is*

$$||\vec{u} - \vec{u}_h||_{\vec{H}(div,\Omega)} + ||p - p_h||_{L^2(\Omega)}$$
$$\leq ch^{r+1}(|p|_{H^{r+1}(\Omega)} + |p|_{H^{r+2}(\Omega)} + |\Delta p|_{H^{r+1}(\Omega)}),$$

for a positive constant c independent of h.

This result implies convergence of the mixed finite element method using the Raviart-Thomas finite element spaces when the continuous problem has sufficiently smooth solutions.

1.5 An extension theorem for Raviart-Thomas finite element spaces.

In this section, we introduce an extension theorem for the Raviart-Thomas velocity spaces, which will be used later to estimate the rate of convergence of various domain decomposition algorithms. By an extension map, we mean a continuous map from a space of boundary values on $\partial\Omega$ to a related space of functions defined on Ω. As before, we let $V_h(\Omega)$ denote the Raviart-Thomas velocity space of order r on a *uniformly shape regular* triangulation τ^h of Ω. Note that the *normal trace* of elements in $V_h(\Omega)$ need not be zero. Let us denote by $V_h(\partial\Omega)$, the space of *normal traces* of elements in $V_h(\Omega)$:

$$V_h(\partial\Omega) \equiv \{\gamma_n \vec{v} : \vec{v}_h \in V_h(\Omega)\}.$$

Thus

$$\gamma_n : V_h(\Omega) \longrightarrow V_h(\partial\Omega),$$

and by applying the *normal trace* lemma to $\vec{v}_h \in V_h(\Omega)$ we obtain:

$$\|\gamma_n \vec{v}_h\|_{H^{-1/2}(\partial\Omega)} \leq C(\Omega)\|\vec{v}_h\|_{\vec{H}(div,\Omega)}.$$

We first note that since γ_n is continuous and onto $V_h(\partial\Omega)$, by an application of the open mapping theorem, γ_n has a continuous right inverse. This right inverse provides an extension of the *normal trace* boundary values in $V_h(\partial\Omega)$ into the Raviart-Thomas velocity space $V_h(\Omega)$. However the norm of this right inverse may depend on h.

When we apply extension theorems to domain decomposition methods to estimate the rate of convergence of these iterative methods, we obtain theoretical bounds which are proportional to the norm of the extension map. Thus, if for each h, we could find an extension E^h, whose norm is bounded independent of h, then our bound for the rate of convergence of the iterative method would be independent of h. In another application, a proof of the uniform *inf-sup* condition for Neumann boundary value problems for mixed formulations of elliptic problems using Raviart-Thomas elements, also requires an extension whose norm is bounded independent of h.

For H^1-conforming finite element functions, such extensions or prolongations from the finite element trace spaces to the finite element spaces, uniformly bounded in h, have been constructed in several ways. See Astrakhantsev [1], Widlund [34], Bjørstad

and Widlund [4], Bramble, Pasciak and Schatz [5], and Matsokin [25]. Using ideas from Bjørstad and Widlund [4], and Bramble, Pasciak and Schatz [5], we construct a uniformly bounded extension for $\vec{H}(div^0,\Omega)$ conforming Raviart-Thomas velocity spaces.

We introduce some notations. Let

$$\begin{array}{rcl} S_h(\Omega) & \equiv & V_h(\Omega)\cap\vec{H}(div^0,\Omega), \\ S_h(\partial\Omega) & \equiv & \{\gamma_n\vec{v}_h : \vec{v}_h\in S_h(\Omega)\}. \end{array}$$

Note, that equivalently, we have

$$S_h(\partial\Omega)=\{g_h\in V_h(\partial\Omega):\int_{\partial\Omega}g_h ds_x=0\}.$$

Before we describe the extension theorem, we state a few Lemmas that will be used in its proof. Let $\widehat{K}$ denote the reference element and $\hat{e}_i\subset\partial\widehat{K}$ one of its edges.

Lemma 12 *Let $0<\epsilon<1$, and $\vec{u}\in(H^\epsilon(\widehat{K}))^2\cap\vec{H}(div^0,\widehat{K})$. Then, $\gamma_n\vec{u}|_{\hat{e}_i}\in H^{\epsilon-1/2}(\hat{e}_i)$. Furthermore*

$$\|\gamma_n\vec{u}\|_{H^{\epsilon-1/2}(\hat{e}_i)}\le c(\epsilon,\widehat{K})\|\vec{u}\|_{(H^\epsilon(\hat{K}))^2},$$

for some positive constant $c(\epsilon,\widehat{K})$.

PROOF OF LEMMA. By an application of the *trace* lemma, since the normal $\vec{n}$ on the edge $\hat{e}_i$ is constant, we obtain that

$$\gamma_n:(H^1(\widehat{K}))^2\longrightarrow H^{1/2}(\hat{e}_i),$$

is a continuous map. Similarly, by an application of the *normal trace* lemma, and the fact that restrictions of $H^{-1/2}(\partial\widehat{K})$ functions to edges $\hat{e}_i\subset\partial\widehat{K}$ are in $(H^{1/2}_{00}(\hat{e}_i))'$, we obtain that

$$\gamma_n:\vec{H}(div^0,\widehat{K})\longrightarrow(H^{1/2}_{00}(\hat{e}_i))',$$

is a continuous map. See the remark in Chapter 4, section 2, for more details about this. Thus, we have that

$$\begin{array}{rcl} \gamma_n:(L^2(\widehat{K}))^2\cap\vec{H}(div^0,\widehat{K}) & \longrightarrow & (H^{1/2}_{00}(\hat{e}_i))', \\ \gamma_n:(H^1(\widehat{K}))^2\cap\vec{H}(div^0,\widehat{K}) & \longrightarrow & H^{1/2}(\hat{e}_i), \end{array}$$

are both continuous maps. Using the theory of interpolation spaces, we obtain that the following map is continuous:

$$\gamma_n:(H^\epsilon(\widehat{K}))^2\cap\vec{H}(div^0,\widehat{K})\longrightarrow H^{\epsilon-1/2}(\hat{e}_i)\quad\text{for } 0<\epsilon<1.$$

i.e., there exists a positive constant $c(\epsilon, \widehat{K})$ such that

$$\|\gamma_n \vec{\hat{u}}\|_{H^{\epsilon-1/2}(\hat{e}_i)} \leq c(\epsilon, \widehat{K})\|\vec{\hat{u}}\|_{(H^\epsilon(\hat{K}))^2}. \square$$

Lemma 13 *Let $\epsilon > 0$ and let Ω be a 2 dimensional polygonal region triangulated by a family of* uniformly shape regular *triangulations $\{\tau^h\}$. Let $\vec{u} \in (H^\epsilon(\Omega))^2 \cap \vec{H}(div^0, \Omega)$, and let Π^h denote the interpolation operator defined onto the Raviart-Thomas finite element velocity space $V_h(\Omega)$. Since $\vec{u}$ is* divergence free, *$\Pi^h \vec{u} \in S_h(\Omega) \equiv V_h(\Omega) \cap \vec{H}(div^0, \Omega)$, and there exists a positive constant $c(\epsilon, \Omega)$ such that:*

$$\|\Pi^h \vec{u}\|_{\vec{H}(div^0,\Omega)} \leq c(\epsilon, \Omega)\|\vec{u}\|_{(H^\epsilon(\Omega))^2}.$$

PROOF OF LEMMA. See the section on Raviart-Thomas elements for the definition of the interpolation map. We first consider the interpolation map on the reference element $\widehat{K}$. By using the representation of the interpolation map as a sum involving linear functionals and its basis, we first check that the linear functionals forming the dual basis are well defined and continuous for $(H^\epsilon(\widehat{K}))^2 \cap \vec{H}(div^0, \widehat{K})$ functions. By Lemma 12, $\gamma_n \vec{\hat{v}}|_{\hat{e}_i} \in H^{\epsilon-1/2}(\hat{e}_i)$. Thus, since

$$L_{i,j}(\vec{\hat{v}}) = \int_{\hat{e}_i} \hat{\phi}_{i,j} \vec{\hat{n}} \cdot \vec{\hat{v}} ds_{\hat{x}},$$

and since $\hat{\phi}_{i,j} \in C^\infty(\hat{e}_i)$, we obtain the bound

$$|L_{i,j}(\vec{\hat{v}})| \leq c(\epsilon, \widehat{K})\|\gamma_n \vec{\hat{v}}\|_{H^{\epsilon-1/2}(\hat{e}_i)}\|\hat{\phi}_{i,j}\|_{H^{1/2-\epsilon}(\hat{e}_i)} \leq c(\epsilon, \widehat{K})\|\vec{\hat{v}}\|_{(H^\epsilon(\hat{K}))^2}\|\hat{\phi}_{i,j}\|_{H^{1/2-\epsilon}(\hat{e}_i)}.$$

Similarly, since

$$L_l(\vec{\hat{v}}) = \int_{\hat{K}} \vec{\wp}_l \cdot \vec{\hat{v}} d\hat{x},$$

and since $\vec{\wp}_l \in (\mathcal{P}_{r-1}(\widehat{K}))^2$, we have

$$|L_l(\vec{\hat{v}})| \leq \|\vec{\hat{v}}\|_{(L^2(\hat{K}))^2}\|\vec{\wp}_l\|_{(L^2(\hat{K}))^2} \leq \|\vec{\hat{v}}\|_{(H^\epsilon(\hat{K}))^2}\|\vec{\wp}_l\|_{(L^2(\hat{K}))^2}.$$

By combining these, we obtain,

$$\|\hat{\Pi}\vec{\hat{v}}\|_{\vec{H}(div,\hat{K})} \leq \sum_{i,j} |L_{i,j}|\|\vec{\hat{v}}\|_{(H^\epsilon(\hat{K}))^2}\|\vec{\hat{u}}_{i,j}\|_{\vec{H}(div,\hat{K})} + \sum_l |L_l|\|\vec{\hat{v}}\|_{(H^\epsilon(\hat{K}))^2}\|\vec{\hat{u}}_l\|_{\vec{H}(div,\hat{K})},$$

i.e.,

$$\|\hat{\Pi}\vec{\hat{v}}\|_{\vec{H}(div,\hat{K})} \leq c(\epsilon, \widehat{K})\|\vec{\hat{v}}\|_{(H^\epsilon(\hat{K}))^2}.$$

Here $c(\epsilon,\widehat{K})$ depends on the sum of the bounds for each linear functional. Thus, $\widehat{\Pi}$ is a bounded map.

Next, we discuss why Π^h is a well defined and bounded map. Recall that, on each element K,

$$(\Pi^h\vec{v})|_K \longleftrightarrow \widehat{\Pi}\hat{\vec{v}},$$

where $\hat{\vec{v}} \longleftrightarrow \vec{v}|_K$. We recall that, by Lemma 6

$$\int_e \phi\gamma_n\vec{v}ds_x = \int_{\hat{e}} \hat{\phi}\gamma_n\hat{\vec{v}}ds_{\hat{x}} = \int_{\hat{e}} \hat{\phi}\gamma_n\widehat{\Pi}\hat{\vec{v}}ds_{\hat{x}}$$

$$= \int_e \phi\gamma_n(\Pi^h\vec{v})ds_x, \quad \forall\phi \in \mathcal{P}_{r-1}(e).$$

Thus, since $\vec{v} \in \vec{H}(div,\Omega)$, and since the interpolant $\gamma_n(\Pi^h\vec{v})|_e \in \mathcal{P}_{r-1}(e)$, we deduce that $\Pi^h\vec{v} \in \vec{H}(div,\Omega)$. To show that the interpolation map Π^h is bounded, we observe that:

1. Using the transformations of semi-norms and norms between K and $\widehat{K}$, as given in Lemma 8, and the boundedness of the reference interpolation map $\widehat{\Pi}$, we obtain:

$$||\Pi^h\vec{v}||_{\vec{H}(div^0,K)} \leq ||\widehat{\Pi}\hat{\vec{v}}||_{\vec{H}(div^0,\hat{K})} \leq c(\epsilon,\widehat{K})||\hat{\vec{v}}||_{(H^\epsilon(\hat{K}))^2},$$

$$\leq c(\epsilon,\widehat{K})(c||\vec{v}||^2_{(L^2(K))^2} + ch_K^\epsilon|\vec{v}|^2_{(H^\epsilon(K))^2})^{1/2} \leq c(\epsilon,\widehat{K})||\vec{v}||_{(H^\epsilon(K))^2},$$

since, $||B_K|| = O(h_K)$; $|J_K| = O(h_K^2)$, etc.

2. Summing up the above result over $\forall K \in \tau^h$, we obtain that $||\Pi^h|| < \infty$. □

Lemma 14 *Let $\epsilon > 0$ and let Ω be a 2 dimensional polygonal region triangulated by a family of* uniformly shape regular *triangulations $\{\tau^h\}$. Let $\vec{u} \in (H^\epsilon(\Omega))^2 \cap \vec{H}(div^0,\Omega)$, and let Π^h denote the interpolation operator defined onto the Raviart-Thomas finite element velocity space $V_h(\Omega)$. Then, there exists a positive constant $c(\epsilon,\Omega)$ such that:*

$$||(I-\Pi^h)\vec{u}||_{\vec{H}(div^0,\Omega)} \leq c(\epsilon,\Omega)h^\epsilon||\vec{u}||_{(H^\epsilon(\Omega))^2}.$$

Here I denotes the identity.

PROOF OF LEMMA. For $\epsilon > 0$, by Lemma 13, we have that Π^h is a bounded linear map

$$\Pi^h : (H^\epsilon(\Omega))^2 \cap \vec{H}(div^0,\Omega) \longrightarrow S_h(\Omega).$$

Using this and a quotient space scaling argument, we will show that

$$\|(I - \Pi^h)\vec{u}\|_{\vec{H}(div^0,\Omega)} \leq c(\epsilon)h^{\epsilon}\|\vec{u}\|_{(H^{\epsilon}(\Omega))^2}.$$

We now present the details of this quotient space argument, which is also used in the study of the convergence of the finite element method. See Ciarlet [12], and Girault and Raviart [15].

Let $\widehat{K}$ denote the reference unit element, a triangle or a square as the case may be, and let K be an element in the triangulation τ^h of Ω. Let F_K denote the affine linear map of $\widehat{K}$ onto K.

$$F_K : \hat{x} \in \widehat{K} \longrightarrow F_K(\hat{x}) = B_K\hat{x} + b_K,$$

where B_K is a 2×2 invertible matrix and b_K is a 2-vector. Let $J_K \equiv det(B_K)$. We use

$$\vec{u} \longleftrightarrow \vec{\hat{u}}$$

to denote the *correspondences* between vector functions on $\widehat{K}$ and on K, as described in an earlier section.

By the relation between corresponding vector functions on K and $\widehat{K}$, we obtain:

$$\|(I - \Pi^h)\vec{u}\|_{(L^2(K))^2} \leq |J_K|^{-1/2}\|B_K\| \ \|(\widehat{I} - \widehat{\Pi})\vec{\hat{u}}\|_{(L^2(\hat{K}))^2}.$$

For $\epsilon > 0$, by the boundedness of $\widehat{I}$ and $\widehat{\Pi}$, we obtain:

$$|J_K|^{-1/2}\|B_K\| \ \|(\widehat{I} - \widehat{\Pi})\vec{\hat{u}}\|_{(L^2(\hat{K}))^2} \leq |J_K|^{-1/2}\|B_K\| \ \|\widehat{I} - \widehat{\Pi}\| \ \|\vec{\hat{u}}\|_{(H^{\epsilon}(\hat{K}))^2}.$$

Since both the identity $\widehat{I}$ and the interpolation map $\widehat{\Pi}$ preserve constants, i.e. elements in $(\mathcal{P}_0(\widehat{K}))^2$, we may replace $\|\vec{\hat{u}}\|_{(H^{\epsilon}(\hat{K}))^2}$ by the quotient space $(H^{\epsilon}(\widehat{K}))^2/(\mathcal{P}_0(\widehat{K}))^2$ norm. By Poincarè's inequality this is bounded by the $(H^{\epsilon}(\widehat{K}))^2$ semi-norm. Thus, we have

$$\|(I - \Pi^h)\vec{u}\|_{(L^2(K))^2} \leq c(\widehat{K})|J_K|^{-1/2}\|B_K\| \ \|\widehat{I} - \widehat{\Pi}\||\vec{\hat{u}}|_{(H^{\epsilon}(\hat{K}))^2}.$$

Now, mapping $\vec{\hat{u}}$ back to $\vec{u}$ on K and using bounds on the Jacobian, we get,

$$|\vec{\hat{u}}|_{(H^{\epsilon}(\widehat{K}))^2} \leq \|B_K\|^{\epsilon}\|B_K^{-1}\| \ |J_K|^{1/2}|\vec{u}|_{(H^{\epsilon}(K))^2}.$$

Since Ω is assumed to be a polygonal domain, we may apply the elliptic regularity theory for this Neumann problem only for $0 \leq \epsilon < 1/2$, see Nečas [27], Grisvard [19], and Babuška and Aziz [3]. If $g_h \in H^{\epsilon-1/2}(\partial\Omega)$ then $\varphi \in H^{1+\epsilon}(\Omega)$. Furthermore we have

$$\|\varphi\|_{H^{1+\epsilon}(\Omega)} \leq c(\Omega,\epsilon)\|g_h\|_{H^{\epsilon-1/2}(\partial\Omega)},$$

for some positive constant $c(\Omega,\epsilon)$. Note, $g_h \in L^2(\partial\Omega)$, therefore $g_h \in H^{\epsilon-1/2}(\partial\Omega)$ since the *normal trace* space $S_h(\partial\Omega)$ consists of piece-wise polynomial functions with discontinuities at the vertices. If Ω were a smooth domain, the elliptic regularity theory would hold for all $\epsilon \geq 0$. For the special case of $\epsilon = 0$ these results can also be obtained by using the standard $H^1(\Omega)$ variational formulation. For details see Nečas [27], Grisvard [19], and Babuška and Aziz [3].

Because $Eg_h = \vec{\nabla}\varphi$, it satisfies $\nabla \cdot Eg_h = 0$. Therefore $Eg_h \in \vec{H}(div^0,\Omega)$. We also have

$$\|Eg_h\|_{(H^{\epsilon}(\Omega))^2} \leq c(\Omega,\epsilon)\|g_h\|_{H^{\epsilon-1/2}(\partial\Omega)},$$

for some positive constant $c(\Omega,\epsilon)$.

However, Eg_h need not be in $S_h(\Omega)$. In order to obtain an extension $E^h g_h$ in the finite element space, we use the interpolation map onto the Raviart-Thomas velocity space. i.e., we let

$$E^h g_h \equiv \Pi^h(Eg_h),$$

where Π^h denotes the Raviart-Thomas velocity interpolation map. See section 1.4, on the Raviart-Thomas finite element spaces for the definition and properties of this interpolation map. Now, we use Lemma 14, which states that:

$$\|(I-\Pi^h)\vec{u}\|_{\vec{H}(div^0,\Omega)} \leq c(\epsilon,\Omega)h^{\epsilon}\|\vec{u}\|_{(H^{\epsilon}(\Omega))^2}.$$

Since

$$\Pi^h Eg_h = Eg_h + (\Pi^h Eg_h - Eg_h),$$

by an application of the triangle inequality, we have

$$\|\Pi^h Eg_h\|_{\vec{H}(div^0,\Omega)} \leq \|Eg_h\|_{\vec{H}(div^0,\Omega)} + c(\epsilon,\Omega))h^{\epsilon}\|Eg_h\|_{(H^{\epsilon}(\Omega))^2}.$$

Using the elliptic regularity results we have

$$\|\Pi^h Eg_h\|_{\vec{H}(div^0,\Omega)} \leq c\|g_h\|_{H^{-1/2}(\partial\Omega)} + c(\epsilon,\Omega)h^{\epsilon}\|g_h\|_{H^{\epsilon-1/2}(\partial\Omega)}.$$

By using the equivalence of all norms on a finite dimensional space, we can bound the $H^{\epsilon-1/2}(\partial\Omega)$ norm of g_h in terms of its $H^{-1/2}(\partial\Omega)$ norm, with a constant possibly depending on h. We estimate this dependence on h, by using an inverse inequality for the finite element space $V_h(\partial\Omega)$. We obtain that

$$\|g_h\|_{H^{\epsilon-1/2}(\partial\Omega)} \le c(\epsilon,\Omega)h^{-\epsilon}\|g_h\|_{H^{-1/2}(\partial\Omega)}. \tag{1.21}$$

Though $V_h(\partial\Omega)$ consists of piece-wise polynomial functions which are *discontinuous* across vertex nodes, we can still prove such an inverse inequality. We outline how this can be shown. First, we prove inequality (1.21) for small $\epsilon > 0$ with $-1/2$ replaced by $+1/2$. Following that, we use duality and interpolation theory to obtain inequality (1.21). For details about this, see Babuška and Aziz [3]. To prove inequality (1.21) with $-1/2$ replaced by $1/2$, we use formula (1.2) to express the norms. Then, we partition $\partial\Omega$ into a disjoint union of edges, each of which is the the edge of some element in the triangulation τ^h. Thus, the norms can be decomposed as as a sum of terms involving the edges that constitute $\partial\Omega$. We then consider the quotient of the semi-norms; this can be estimated by mapping the numerator and denominator onto a reference unit element by using an affine linear map. Since g_h restricted to any edge is a polynomial, we can bound the quotient on the reference element by a constant depending only on the dimension of the polynomial space of the reference edge, and the quotient of semi-norms on this polynomial space. Using the bounds for the Jacobian of the map between the edge and a reference element, in terms of h, we can establish inequality (1.21) with $-1/2$ replaced by $+1/2$. We do not present the details. See Babuška and Aziz [3], Ciarlet [12] and Girault and Raviart [15]. Putting these results together, we obtain

$$\|\Pi^h E g_h\|_{\vec{H}(div^0,\Omega)} \le c\|g_h\|_{H^{-1/2}(\partial\Omega)} + c(\epsilon,\Omega)h^{\epsilon}h^{-\epsilon}\|g_h\|_{H^{-1/2}(\partial\Omega)}.$$

Thus, $E^h = \Pi^h E$ is bounded independently of h

$$\|\Pi^h E g_h\|_{\vec{H}(div^0,\Omega)} \le c(\epsilon,\Omega)\|g_h\|_{H^{-1/2}(\partial\Omega)}.$$

Note that by construction, $\Pi^h E g_h \in S_h(\Omega)$, and $\gamma_n E g_h = g_h$. Thus, $E^h \equiv \Pi^h E$ is a uniformly bounded extension. □

1.6 Iterative methods for solving symmetric, positive definite linear systems.

Let A be an real, $n \times n$ symmetric, positive definite matrix, and $b \in R^n$. We consider solving linear systems of the following form

$$\text{Find } x \in R^n, \text{ such that } \quad Ax = b,$$

using iterative methods.

Let M be a symmetric, positive definite matrix, which is easier to invert than A, in the sense of solving $Mx = b$. For $i = 0, 1, 2, \ldots$, given the iterate x^i, the next iterate, x^{i+1} is defined by

$$Mx^{i+1} = (M - A)x^i + b. \tag{1.22}$$

The initial iterate, x^0 is arbitrary and can be chosen to be zero. We note that equation (1.22), when considered as a mapping from x^i to x^{i+1}, has a fixed point at x, i.e., x is the unique solution of

$$Mx = (M - A)x + b. \tag{1.23}$$

Subtracting equation (1.22) from equation (1.23), we obtain,

$$M(x - x^{i+1}) = (M - A)(x - x^i).$$

Letting $e^i \equiv x - x^i$, denote the error in the iterate x^i, we obtain $Me^{i+1} = (M - A)e^i$, i.e.,

$$e^{i+1} = (I - M^{-1}A)e^i.$$

Let $\rho(I - M^{-1}A)$ denote the spectral radius of $I - M^{-1}A$. Using the eigenfunction expansion of e^0, we see that if $\rho(I - M^{-1}A) < 1$, the iterative method converges. We note that $I - M^{-1}A$ is diagonalisable since it is similar to $M^{1/2}(I - M^{-1}A)M^{-1/2} = (I - M^{-1/2}AM^{-1/2})$, which is a symmetric matrix. In the rest of the thesis, we consider iterative methods to solve the saddle point problems obtained from the mixed finite element discretisation of elliptic problems using the Raviart-Thomas spaces.

For various choices of M, the spectral radius $\rho(I - M^{-1}A)$ may not be less than one. In such cases, we can use an "acclerated" iterative method, such as the conjugate

gradient method, to obtain convergence. We may also use the acclerated methods even when the basic iterative method is convergent. We breifly describe the preconditioned conjugate gradient algorithm, and then present the error propagation results. For details, see Axelsson and Barker [2], Golub and Van Loan [17], and Hageman and Young [20]. The algorithm is described for the symmetric, positive definite linear system $Ax = b$, with a symmetric, positive definite preconditioner M. Let $\epsilon > 0$ denote a tolerance used as a stopping criterion, and $\tilde{x}$ the iterate obtained when the stopping criterion is satisfied.

$x^0 \equiv 0$
$r^0 \equiv b$
For $i = 1, 2, \ldots$
    if $||r^{i-1}|| \leq ||r^0||\epsilon$ *then*
        Set $\tilde{x} = x^{i-1}$ and quit.
    else
        Solve $Mz^{i-1} = r^{i-1}$ for z^{i-1}.
        $\beta^i = [z^{i-1}, r^{i-1}]/[z^{i-2}, r^{i-2}]$ $(\beta^1 \equiv 0)$
        $p^i = z^{i-1} + \beta^i p^{i-1}$ $(p^1 \equiv z^0)$
        $\alpha^i = [z^{i-1}, r^{i-1}]/[p^i, Ap^i]$
        $x^i = x^{i-1} + \alpha^i p^i$
        $r^i = r^{i-1} - \alpha^i Ap^i$
endfor.

Here $[x, y]$ denotes the euclidean inner product $x^T y$, when A and M are symmetric in the usual sense. If $M = I$, and A is symmetric in an inner product which is not the euclidean inner product, then $[.,.]$ will denote that inner product.

To obtain a computationally more efficient algorithm, we assume that it is easier to invert the preconditioner M than the matrix A, in the sense of solving linear systems; i.e., it is computationally easier to solve $Mz = d$, than to solve $Az = d$. Furthermore, we assume that the *spectral condition number* of $M^{-1/2}AM^{-1/2}$ is smaller than the *spectral condition number* of A. The *spectral condition number* of a nonsingular matrix B is defined to be $||B||||B^{-1}||$, where $||B||$ and $||B^{-1}||$ denote the euclidean norm of B and B^{-1} respectively. Note that the algorithm only involves use of the product Ax, and the solution of $My = z$, and other vector and scalar operations. In particular, we

do not need to assemble the matrix A or M, we only need to know how to compute Ax and $M^{-1}z$.

The error for the conjugate gradient method with $M = I$ is bounded by

$$||x - x^i||_A \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^i ||x - x^0||_A,$$

where the A-norm $||w||_A \equiv (w^T A w)^{1/2}$, and $\kappa = \kappa(A)$ is the spectral condition number of A. A similar result holds for the preconditioned conjugate gradient method with $\kappa(A)$ replaced by $\kappa(M^{-1/2}AM^{-1/2})$. Note that the spectral condition number of A is equivalently defined by λ_2/λ_1, where

$$\lambda_1 \leq \frac{x^T A x}{x^T x} \leq \lambda_2,$$

are the maximum and minimum eigenvalues of A given by the extremum of the Rayleigh quotient of A. Similarly, the spectral condition number of $M^{-1/2}AM^{-1/2}$ is equivalently given by λ_2/λ_1, the quotient of the extremum of the generalised Rayleigh quotient

$$\lambda_1 \leq \frac{x^T A x}{x^T M x} \leq \lambda_2.$$

Finally, we remark that if $M = I$, and if A is symmetric, positive definite in an inner product $[.,.]$ which may be different from the Euclidean inner product, then the same conjugate gradient algorithm holds, with use of the $[.,.]$ inner product.

Chapter 2

The Schwarz methods for Raviart-Thomas elements.

In this chapter, we introduce the multiplicative and additive Schwarz methods which are iterative methods to solve symmetric positive definite problems. First, we describe these methods in a Hilbert space. Following that, we outline their application to saddle point problems. We then study in detail the application of these methods to the particular case of mixed finite element discretisations of elliptic problems with Neumann boundary conditions using the Raviart-Thomas spaces. Finally, we present the results of various numerical experiments conducted using these methods.

2.1 The Schwarz alternating methods on a Hilbert space.

Let $\mathcal{H}$ be a finite dimensional Hilbert space with inner product $a(.,.)$, and let $f(.)$ be a continuous linear functional on $\mathcal{H}$. Consider the following problem:

$$\begin{cases} \text{Find } u \in \mathcal{H} \text{ such that} \\ a(u,v) = f(v), \quad \forall v \in \mathcal{H}. \end{cases} \tag{2.1}$$

In matrix language this is just a symmetric, positive definite linear system, and it could be solved by a standard iterative method such as the conjugate gradient method. We first describe the Schwarz alternating method, or as we will refer to it, the multiplicative Schwarz method.

2.1.1 The multiplicative Schwarz method.

The multiplicative Schwarz method is an iterative method to solve problem (2.1), cf. Lions [22]. We assume that the space $\mathcal{H}$ is decomposed as a sum of smaller subspaces. Then, during each iteration, which usually consists of a few fractional steps, each of which involves a defect correction on one of the subspaces, we update the old iterate so that the iterate in the current fractional step has zero residual with respect to the appropriate subspace. Thus, each iteration involves the solution of problems of a smaller size and often some subproblems can be solved concurrently, i.e., in parallel.

We now describe this in more detail. For $i = 0, \dots, n$, let $\mathcal{H}_i$ be subspaces of $\mathcal{H}$ that sum to $\mathcal{H}$:

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \cdots + \mathcal{H}_n.$$

Let P_i denote the orthogonal projection onto $\mathcal{H}_i$, in the $a(.,.)$ inner product. Thus $P_i u \in \mathcal{H}_i$ satisfies

$$a(P_i u, v) = a(u, v) \quad \forall v \in \mathcal{H}_i. \tag{2.2}$$

Finding $P_i u$ is equivalent to solving a "subproblem" on $\mathcal{H}_i$.

Let $u^0 \in \mathcal{H}$ be an arbitrarily chosen initial iterate. Given the ith iterate, u^i, for $i = 0, 1, 2, \dots$, we compute the next iterate, u^{i+1}, in $n+1$ fractional steps. Each fractional step consists of a defect correction on one of the subspaces. i.e., for $j = 0, \dots, n$, let $u^{i+\frac{j+1}{n+1}} \in \mathcal{H}$ be sequentially defined by:

$$\begin{cases} u^{i+\frac{j+1}{n+1}} \equiv u^{i+\frac{j}{n+1}} + \delta u^{i+\frac{j}{n+1}}, \\ \text{where } \delta u^{i+\frac{j}{n+1}} \in \mathcal{H}_j \text{ is chosen so that} \\ a(u^{i+\frac{j+1}{n+1}}, v) = f(v), \quad \forall v \in \mathcal{H}_j. \end{cases}$$

This step is known as the defect correction on $\mathcal{H}_j$. Thus, $\delta u^{i+\frac{j}{n+1}} = P_j(u - u^{i+\frac{j}{n+1}})$, is the solution of

$$a(\delta u^{i+\frac{j}{n+1}}, v) = a(u - u^{i+\frac{j}{n+1}}, v) = f(v) - a(u^{i+\frac{j}{n+1}}, v) \quad \forall v \in \mathcal{H}_j. \tag{2.3}$$

Since the subproblems are of a smaller size, they may be solved with more ease than the full problem. In case some subspaces $\{\mathcal{H}_j\}$ are mutually $a(.,.)$-orthogonal, then we can solve some of these subproblems concurrently, i.e., in parallel. We will discuss this in the applications. Note that the right hand side in equation (2.3) can be computed without knowing u, since $a(u,.) = f(.)$ is given.

The error, $e^i \equiv u - u^i$, is found to satisfy

$$e^{i+1} = P_k^{\perp} P_{k-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp} e^i,$$

where $P_j^{\perp} = I - P_j$ denotes the $a(.,.)$-orthogonal projection onto $\mathcal{H}_j^{\perp}$. Thus, we have

$$\|e^i\| \leq \rho^i \|e^0\|, \quad \text{where } \rho \equiv \|P_k^{\perp} P_{k-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp}\|,$$

and the norm is defined by

$$\|w\| \equiv \sqrt{a(w,w)}.$$

Since orthogonal projections have norms bounded by one, $\rho \leq 1$. However, for convergence of this iterative method, we need $\rho < 1$.

The following Lemmas, discuss sufficient conditions for the convergence of the multiplicative Schwarz method. The main condition is that the subspaces $\mathcal{H}_j$ sum to $\mathcal{H}$. However, explicit bounds for the convergence factor, ρ, can be obtained in terms of other parameters.

The next Lemma establishes certain inequalities satisfied by the projections. It is also used in the proof of convergence of the additive Schwarz method. It is given in Lions [22] for the case k=2. Also see Dryja and Widlund [13].

Lemma 15 (Lions) *Let $\mathcal{H}$ be a Hilbert space with inner product $a(.,.)$, such that $\mathcal{H} = \mathcal{H}_0 + \cdots + \mathcal{H}_n$, where $\mathcal{H}_0, \ldots, \mathcal{H}_n$ are subspaces. For $v \in \mathcal{H}$, suppose there exists a partition $v = v_0 + \cdots + v_n$, with $v_i \in \mathcal{H}_i$, satisfying*

$$\sum_{i=0}^{n} a(v_i, v_i) \leq \mu^2 a(v,v),$$

for a positive constant μ. Then

$$a(v,v) \leq \mu^2 \sum_{i=0}^{n} a(P_i v, P_i v) = \mu^2 \sum_{i=0}^{n} a(P_i v, v) = \mu^2 a(Pv, v), \tag{2.4}$$

where $P \equiv P_0 + \cdots + P_n$, and the P_i are the orthogonal projections onto $\mathcal{H}_i$ in the $a(.,.)$ inner product.

PROOF OF LEMMA. Let $v = \sum_{i=0}^{n} v_i$ be a decomposition satisfying the above hypothesis. Then

$$a(v,v) = \sum_{i=0}^{n} a(v_i, v) = \sum_{i=0}^{n} a(P_i v, v_i).$$

By using the Cauchy-Schwarz inequality, we obtain

$$a(v,v) \le (\sum_{i=0}^{n} a(P_i v, P_i v))^{1/2} (\sum_{i=0}^{n} a(v_i, v_i))^{1/2}.$$

By using the hypothesis, we obtain

$$a(v,v) \le \mu(\sum_{i=0}^{n} a(P_i v, v))^{1/2} (a(v,v))^{1/2}.$$

Squaring both sides and cancelling the common terms gives us

$$a(v,v) \le \mu^2 \sum_{i=0}^{n} a(P_i v, v) = \mu^2 a(Pv, v). \square$$

The next Lemma introduces sufficient conditions for convergence of the multiplicative Schwarz method. However, no estimate for the convergence factor ρ is given.

Lemma 16 *Let $\mathcal{H}$ be a finite dimensional Hilbert space with inner product $a(.,.)$ and subspaces $\mathcal{H}_i$ that sum to $\mathcal{H}$:*

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \cdots + \mathcal{H}_n.$$

Then:
(1) there exists a positive constant c_0, such that, for any $u \in \mathcal{H}$, there exists a partition $u = \sum_{i=0}^{n} u_i$, with

$$a(u,u) \le c_0^2 \sum_{i=0}^{n} a(P_i u, P_i u), \tag{2.5}$$

where, as before, P_i denotes the orthogonal projection onto $\mathcal{H}_i$, in the $a(.,.)$ inner product.
(2) there exists a positive constant $\rho < 1$ such that the error propagation map for the multiplicative Schwarz method satisfies

$$\|P_n^{\perp} P_{n-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp}\| \equiv \rho < 1.$$

PROOF OF LEMMA. We follow the proof given in Lions [22]. Consider the following map:

$$(u_0, \ldots, u_n) \in \mathcal{H}_0 \times \cdots \times \mathcal{H}_n \longrightarrow (u_0 + \cdots + u_n) \in \mathcal{H}.$$

Let the norm on the product space be defined by

$$\|(u_0, \ldots, u_k)\| \equiv (\sum_{i=0}^{n} \|u_i\|^2)^{1/2},$$

then, clearly, the addition map is a continuous map, and it is surjective by hypothesis. Thus, by an application of the open mapping theorem, there exists a continuous right inverse from $\mathcal{H}$ to the product space. i.e., given $u \in \mathcal{H}$, there exists a partition $u = \sum_{i=0}^{n} u_i$, with $u_i \in \mathcal{H}_i$, and satisfying

$$\sum_{i=0}^{n} a(u_i, u_i) \leq c_0^2 a(u, u)$$

for some positive constant c_0. By applying Lemma 15, we obtain equation (2.5).

To prove the second part, we first note that since each of the projections are orthogonal, their norms are equal to one. Thus $\rho \leq 1$. To obtain a contradiction, suppose that $\rho = 1$. Then, since $\mathcal{H}$ is a finite dimensional Hilbert space, the maximum of the quotient defining the norm of $P_n^{\perp} P_{n-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp}$, is attained at some point u on the unit sphere in $\mathcal{H}$. i.e.,

$$||P_n^{\perp} P_{n-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp} u|| = ||u|| = 1.$$

Thus, we must have

$$||P_0^{\perp} u|| = 1 = ||u|| \Longrightarrow P_0 u = 0, \text{ and } P_0^{\perp} u = u.$$

By induction, we obtain that

$$||P_j^{\perp} \cdots P_0^{\perp} u|| = ||P_j^{\perp} u|| = 1 = ||u||, \Longrightarrow P_j u = 0, \quad \forall j.$$

But this contradicts equation (2.5). Thus, our assumption that $\rho = 1$ is wrong. Our result now follows. □

The following result concerns the special case of the convergence of the multiplicative Schwarz method in the case of two subspaces. In this case, we are able to estimate the convergence factor ρ in terms of the parameter c_0, cf. Lions [22].

Lemma 17 *Let $\mathcal{H}$ be a Hilbert space with inner product $a(.,.)$ and subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$ that sum to $\mathcal{H}$. Assume that there exists a a positive constant c_0 such that*

$$\forall u \in \mathcal{H}, \quad ||u|| \leq c_0(||P_1 u||^2 + ||P_2 u||^2)^{1/2}.$$

Then,

$$\rho = ||P_2^{\perp} P_1^{\perp}|| \leq (1 - \frac{1}{c_0^2})^{1/2}.$$

PROOF OF LEMMA. Applying our hypothesis to $u = P_1^{\perp} v$, we obtain

$$\|P_1^{\perp} v\| \leq c_0(\|P_1 P_1^{\perp} v\|^2 + \|P_2 P_1^{\perp} v\|^2)^{1/2} = c_0 \|P_2 P_1^{\perp} v\|.$$

By using the Pythagorian theorem, we obtain

$$\begin{aligned} \|P_1^{\perp} v\|^2 &= \|P_2 P_1^{\perp} v\|^2 + \|P_2^{\perp} P_1^{\perp} v\|^2 \\ &\geq \tfrac{1}{c_0^2}\|P_1^{\perp} v\|^2 + \|P_2^{\perp} P_1^{\perp} v\|^2 \Longrightarrow \\ \|P_2^{\perp} P_1^{\perp} v\| &\leq (1 - \tfrac{1}{c_0^2})^{1/2} \|P_1^{\perp} v\| \\ &\leq (1 - \tfrac{1}{c_0^2})^{1/2} \|v\|. \end{aligned}$$

Since, v is arbitrary, the result follows. □

Many applications of the multiplicative Schwarz method involve more than two subspaces. The following result, which is an extension of Lemma 16 due to Lions [22], for the many subspace multiplicative Schwarz method, presents an estimate for the convergence factor ρ in terms of other parameters, such as c_0, that can be estimated. However, the result gives a pessimistic and poorer estimate than Lemma 17. Following that, we present a result for the *symmetrised* multiplicative Schwarz method. See Widlund [35].

Lemma 18 *Let $\mathcal{H}$ be a Hilbert space with inner product $a(.,.)$. Let $\mathcal{H}_0, \ldots, \mathcal{H}_n$ be subspaces of $\mathcal{H}$ that sum to $\mathcal{H}$. Suppose that there exists a positive constant c_0 such that for all $u \in \mathcal{H}$, there exists a partition $\{u_i \in \mathcal{H}_i\}$ with*

$$u = u_0 + \cdots + u_n,$$

satisfying

$$\sum_{i=0}^{n} a(u_i, u_i) \leq c_0^2 a(u, u). \tag{2.6}$$

Then, the convergence factor of the multiplicative Schwarz method

$$\rho = \|P_n^{\perp} \cdots P_0^{\perp}\|,$$

satisfies

$$\rho^2 \leq 1 - \left(\frac{1}{d_n c_0^2}\right)^{2^{n-1}}.$$

Here $1 \leq d_n$, is given by

$$d_n = \sum_{i=1}^{n} [1 + i(i+1)].$$

PROOF. We first prove the result for finite dimensional Hilbert spaces. Since the unit ball in any finite dimensional Hilbert space is compact, there exists an element u on the unit sphere of $\mathcal{H}$ such that

$$\rho = \|P_n^\perp \cdots P_0^\perp u\|.$$

Since we have a product of orthogonal projections,

$$0 \leq \rho \leq 1.$$

It follows that,

$$\|P_i^\perp \cdots P_0^\perp u\| \geq \rho, \qquad \text{for } 0 \leq i \leq n.$$

For $i = 0$,

$$\|P_0^\perp u\| = 1 \Longrightarrow P_0 u = 0, \;\; P_0^\perp u = u.$$

This is so because, we can write $u = P_0 u + P_0^\perp u$, and

$$\rho^2 = \frac{\|P_n^\perp \cdots P_0^\perp u\|^2}{\|P_0 u\|^2 + \|P_0^\perp u\|^2}.$$

The numerator is independent of $P_0 u$, and the denominator can be reduced if $P_0 u \neq 0$. Thus, ρ can be increased unless $P_0 u = 0$.

Next, We next show that for $i \geq 1$

$$\|P_i u\|^2 \leq 1 - \rho^2 + 2(\sum_{j=0}^{i-1} \|P_j u\|).$$

To do this, we consider,

$$\|P_i u\|^2 = 1 - \|P_i^\perp u\|^2 = 1 - \|P_i^\perp(P_{i-1}^\perp \cdots P_0^\perp)u + P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u\|^2$$

$$= 1 - (\|P_i^\perp \cdots P_0^\perp u\|^2 + 2a(P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u, P_i^\perp P_{i-1}^\perp \cdots P_0^\perp u) + \|P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u\|^2)$$

$$\leq 1 - \|P_i^\perp \cdots P_0^\perp u\|^2 + 2\|P_i^\perp \cdots P_0^\perp u\| \|P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u\|$$

$$\leq 1 - \rho^2 + 2\|P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u\|.$$

Since

$$\|P_i^\perp(I - P_{i-1}^\perp \cdots P_0^\perp)u\| \leq \|(I - P_{i-1}^\perp \cdots P_0^\perp)u\|,$$

we consider a bound for the right hand side, which we write as,

$$(I - P_{i-1}^\perp \cdots P_0^\perp)u = (I + P_{i-1}^\perp \cdots P_1^\perp P_0)u - P_{i-1}^\perp \cdots P_1^\perp u$$

$$= (I + P_{i-1}^{\perp} \cdots P_1^{\perp} P_0 + \cdots + P_{i-1}^{\perp} P_{i-2} - P_{i-1}^{\perp})u.$$

Using the triangle inequality, we obtain

$$\begin{aligned} \|(I - P_{i-1}^{\perp} \cdots P_0^{\perp})u\| &\le \|P_{i-1}u + \cdots + P_{i-1}^{\perp} \cdots P_1^{\perp} P_0 u\| \\ &\le \sum_{j=0}^{i-1} \|P_j u\|. \end{aligned}$$

Thus, our bound for $\|P_i u\|$ is

$$\|P_i u\|^2 \le 1 - \rho^2 + 2(\sum_{j=0}^{i-1} \|P_j u\|), \qquad \text{for } i \ge 1.$$

We have already shown that

$$\|P_0 u\| = 0.$$

Denoting $r_i \equiv \|P_i u\|$, we obtain that,

$$\begin{cases} r_i^2 \le 1 - \rho^2 + 2(r_0 + \cdots + r_{i-1}) & \text{for } i \ge 1, \\ r_0^2 = 0 & \text{for } i = 0. \end{cases} \tag{2.7}$$

Using equation (2.6), equation (2.4), and our definition of r_i, we obtain that

$$1 \le c_0^2 \sum_{i=0}^{n} r_i^2.$$

Since each $r_i \longrightarrow 0$, as $\rho \longrightarrow 1$, the right hand side approaches zero as $\rho \longrightarrow 1$. This would lead to a contradiction, since the left handside is 1. Thus, ρ must be bounded below by a number less than one, which depends on c_0. We now give a pessimistic estimate for ρ, using square roots in equation (2.7), and the fact that $\sqrt{1-\rho^2}$ is greater than $1 - \rho^2$, for $0 < \rho < 1$:

$$\begin{cases} r_0^2 = 0 \\ r_1^2 \le 1 - \rho^2 \\ r_2^2 \le 1 - \rho^2 + 2\sqrt{1-\rho^2} & \le 3\sqrt{1-\rho^2} \\ r_3^2 \le 1 - \rho^2 + 2(\sqrt{1-\rho^2} + \sqrt{1 - \rho^2 + 2\sqrt{1-\rho^2}}) & \le \cdots \\ \quad \vdots \end{cases}$$

from which we obtain by induction that

$$\frac{1}{c_0^2} \le \left(1 - \rho^2\right)^{(\frac{1}{2})^{n-1}} \sum_{i=1}^{n} [1 + i(i+1)],$$

and that

$$\rho^2 \le 1 - \left(\frac{1}{c_0^2 \sum_{i=1}^{n} [1 + i(i+1)]}\right)^{2^{n-1}}.$$

To prove the result for infinite dimensional spaces, we use a sequence of vectors on the unit ball such that the norm of the error propagation map acting on the sequence approaches the limit ρ as the sequence approaches its limit. We then apply the same arguments as for the finite dimensional case and obtain the result. □

DEFINITION. An iteration of the *symmetrised* multiplicative Schwarz method consists of $2n+1$ fractional steps rather than the $n+1$ fractional steps for the standard multiplicative Schwarz method. They involve defect correction on

$$\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{n-1}, \mathcal{H}_n, \mathcal{H}_{n-1}, \ldots, \mathcal{H}_1, \mathcal{H}_0,$$

in that order.

The error propagation map associated with the *symmetrised* multiplicative Schwarz method is

$$e^{i+1} = E_n^T E_n e^i; \qquad \text{where } E_n \equiv P_n^{\perp} P_{n-1}^{\perp} \cdots P_1^{\perp} P_0^{\perp}.$$

Thus, the error propagation map is symmetric in the $a(.,.)$ inner product, and therefore, the *symmetrised* multiplicative Schwarz method to solve problem (2.1) can be acclerated using the conjugate gradient method.

The appropriate quantity determining the rate of convergence of the *symmetrised* multiplicative Schwarz method is the norm of the error propagation operator, $\|E_n^T E_n\|$, which, by use of the previous lemmas, is less than one. The quantity determining the rate of convergence of the preconditioned conjugate gradient method is the quotient of extreme eigenvalues of $I - E_n^T E_n$. An upper bound for the spectra of $I - E_n^T E_n$ is 1, since $E_n^T E_n$ is a positive semi-definite map, in the $a(.,.)$ inner product. A lower bound for the spectra of $I - E_n^T E_n$ is $1 - \rho(E_n^T E_n) > 0$, where $\rho(E_n^T E_n) < 1$ is the spectral radius of $E_n^T E_n$. Thus, the condition number of the preconditioned *symmetrised* multiplicative Schwarz method is given by $1/(1 - \rho(E_n^T E_n))$.

Thus, if we have a bound for the condition number $\kappa(I - E_n^T E_n)$, we obtain

$$\rho(E_n^T E_n) = 1 - \frac{1}{\kappa(I - E_n^T E_n)}, \tag{2.8}$$

and this in turn gives an estimate for the spectral radius of the *unsymmetrised* multiplicative Schwarz method:

$$\rho(E_n) \leq \sqrt{1 - \frac{1}{\kappa(I - E_n^T E_n)}}. \tag{2.9}$$

The following Lemma, stated in Widlund [35], gives an upper bound for the condition number of the *symmetrised* Schwarz method involving $n+1$ subspaces, in terms of the product of the condition numbers of n 2 subspace *symmetrised* Schwarz methods. Let $\kappa(\mathcal{H}_0, \ldots, \mathcal{H}_n)$ denote the condition number of the *symmetrised* multiplicative Schwarz method. Also, we use $\kappa(\mathcal{H}_j, \sum_{i=0}^{j-1} \mathcal{H}_i)$ to denote the condition number of the *symmetrised* multiplicative Schwarz method on $\sum_{i=0}^{j} \mathcal{H}_i$ using subspaces $\mathcal{H}_j$ and $\sum_{i=0}^{j-1} \mathcal{H}_i$ respectively. Then:

Lemma 19 *The condition number for the* symmetrised *multiplicative Schwarz method satisfies*

$$\begin{aligned} &\kappa(\mathcal{H}_0, \mathcal{H}_1, \ldots, \mathcal{H}_{n-1}, \mathcal{H}_n) \\ &\leq \kappa(\mathcal{H}_n, \textstyle\sum_{i=0}^{n-1} \mathcal{H}_i)\ \kappa(\mathcal{H}_{n-1}, \sum_{i=0}^{n-2} \mathcal{H}_i) \cdots \kappa(\mathcal{H}_2, \mathcal{H}_1 + \mathcal{H}_0)\ \kappa(\mathcal{H}_1, \mathcal{H}_0), \end{aligned} \tag{2.10}$$

where the terms on the right hand side involve the condition numbers of two subspace symmetrised *multiplicative Schwarz method using the subspaces indicated.*

The advantage here is that we can obtain explicit bounds for the condition numbers of the two subspace cases. We can estimate each term in the right hand side of equation (2.10) by using equation (2.8) to estimate the condition number of the two subspace *symmetrised* methods in terms of the spectral radius of the *symmetrised* method. The spectral radius of the 2 subspace *symmetrised* methods can often, by using lemma 17, be shown to be independent of mesh parameters, etc. Applying Lemma 19, and equation (2.9), we obtain that

$$\rho(E_n)^2 \leq 1 - \frac{1}{\kappa_n \cdots \kappa_1},$$

where $\kappa_j \equiv \kappa(\mathcal{H}_j, \sum_{i=0}^{j-1} \mathcal{H}_i)$. We present applications of this, later in this Chapter and in the Chapter on iterative refinement methods. We present a proof of Lemma 19 in the special case of iterative refinement methods, in the next Chapter.

2.1.2 The additive Schwarz method.

The additive version of the Schwarz alternating method, as described by Dryja and Widlund [13], is based upon the multiplicative Schwarz method, but modified so that the resulting algorithm is more parallelisable. It is an iterative method to solve equation (2.1), and it involves the concurrent solution of subproblems during each

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iteration, rather than the sequential solution of subproblems during each iteration as in the case of the multiplicative Schwarz method. The method is based upon a transformation of the given problem to a new, well conditioned linear system with the same solution. Recall that in the multiplicative Schwarz algorithm, each iteration involves either $n + 1$ or $2n + 1$ fractional steps, depending on whether the *unsymmetrised* or *symmetrised* version is used. These fractional steps are performed sequentially, although in some applications, as we shall discuss in the section on applications to mixed finite element discretisations of elliptic Neumann problems, many fractional steps can be performed concurrently. Thus, the additive Schwarz method is more parallelisable within each iteration. However, the convergence rate for the additive Schwarz method may be slower than for the multiplicative Schwarz method, and the overall performance may depend on the specific problem being solved, and the architecture of the computer being used, etc.

We briefly describe the additive Schwarz method. Define P by

$$P \equiv P_0 + P_1 + \ldots + P_n. \tag{2.11}$$

Note that since P is a sum of $a(.,.)$-orthogonal projections, P is symmetric in the $a(.,.)$ inner product. Now, given $f(.)$, we can compute $g = Pu$, where u is the solution of problem (2.1), without knowing u, by replacing $a(u,v)$ by $f(v)$ in equation (2.2). If P is positive definite in the $a(.,,)$ inner product, then we could use an iterative method, such as the conjugate gradient method, to solve $Pu = g$ for u. Note that, since the action of P is the sum of various projections, its action is highly parallelisable. The main abstract result concerning the additive Schwarz method is given in the following lemma. See Dryja and Widlund [13].

Lemma 20 *As before, let $\mathcal{H}$ be a Hilbert space with inner product $a(.,.)$, and let $\mathcal{H}_i$ be subspaces that sum to $\mathcal{H}$:*

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1 + \cdots + \mathcal{H}_n.$$

We let P_i denote the $a(.,.)$ orthogonal projection onto $\mathcal{H}_i$, and

$$P \equiv P_0 + P_1 + \cdots + P_n.$$

Then,

$$\forall u \in \mathcal{H}, \quad a(Pu,u) \leq (n+1)a(u,u).$$

Suppose that there exists a positive constant c_0 such that $\forall u \in \mathcal{H}$, there exists a partition

$$u = u_0 + u_1 + \cdots + u_n,$$

with $u_i \in \mathcal{H}_i$ satisfying

$$\sum_{i=0}^{n} a(u_i, u_i) \leq c_0^2 a(u, u),$$

then

$$\forall u \in \mathcal{H}, \quad \frac{1}{c_0^2} a(u, u) \leq a(Pu, u).$$

PROOF OF LEMMA. By using the definition of P as a sum of $n+1$ $a(.,.)$ orthogonal projections, we obtain

$$a(Pu, u) = \sum_{i=0}^{n} a(P_i u, u) = \sum_{i=0}^{n} a(P_i u, P_i u) \leq \sum_{i=0}^{n} a(u, u) = (n+1) a(u, u).$$

We use our hypothesis and Lemma 16 to obtain the lower bound for P in the $a(.,.)$ inner product:

$$\frac{1}{c_0^2} a(u, u) \leq a(Pu, u).$$

This shows that P is positive definite in the $a(.,.)$ inner product. The rate of convergence of the conjugate gradient method to solve $Pu = g$, is bounded in terms of the condition number of P which is bounded by $(n+1)c_0^2$. □

Dryja and Widlund, see [13], have studied this algorithm in the case of standard finite element discretizations of elliptic problems, in 2 and 3 dimensions. They show that in the case where one of the subspaces is a coarse model, and the other subspaces are suitably chosen, then there exists positive constants δ_1, δ_2, independent of the mesh parameters, such that

$$\delta_1 \, a(u, u) \leq a(Pu, u) \leq \delta_2 \, a(u, u), \quad \forall u \in S.$$

Thus, if the conjugate gradient method is used to solve $Pu = g$, in the $a(.,.)$ inner product, the number of iterations would depend on δ_2/δ_1, which is independent of the mesh parameters. Numerical tests run by Greenbaum, Li and Chao [18] verify the theoretical results obtained.

We also mention that X. C. Cai has studied a generalisation of the additive Schwarz method to certain non-symmetric problems, including convection diffusion problems, and time-dependent diffusion problems. See Cai [11].

2.1.3 The Schwarz methods for saddle point problems.

We now consider the steps involved in applying iterative methods to saddle point problems. In particular, we consider the multiplicative and additive Schwarz methods. See also Lions [22], and Glowinski and Wheeler [16]. Consider the discrete saddle point problem obtained by discretising the continuous problem (1.7).
Find $u_h \in X_h, p_h \in Y_h$ such that:

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_h \\ p_h \end{pmatrix} = \begin{pmatrix} W_h \\ F_h \end{pmatrix}, \tag{2.12}$$

where A_h is a symmetric, positive definite matrix, and B_h is of full rank. With some abuse of notation, we use u_h, p_h, etc., to denote functions as well as their vector representation in matrix notation.

Note that, for discretisation of elliptic Neumann boundary value problems, p_h is unique only in $X_h(\Omega) = Q_h(\Omega)/R$, i.e., unique up to a constant in $Q_h(\Omega)$. Thus, if we use a basis for $Q_h(\Omega)$ in our construction of the matrix, then B_h^T will have a null space spanned by $(1, \ldots, 1)^T$. We refer to functions in X_h as velocities, and functions in Y_h as pressures; B_h will be refered to as the *discrete divergence* map.

Let

$$\mathcal{H} \equiv \{u_h \in X_h : B_h u_h = 0\}.$$

Elements in $\mathcal{H}$ will be denoted with a subscript DF, for example $v_{h,DF}$ is *discrete divergence free*. We can decompose, in many ways, the velocity space

$$X_h = X_{h,I} + \mathcal{H};$$

$X_{h,I}$ can even be taken to be X_h. Let $X_0, \ldots, X_n$ be subspaces of X_h that sum to X_h. Let $Y_0, \ldots, Y_n$ be the ranges of the B_h map acting on $X_0, \ldots, X_n$ respectively. Let, $\mathcal{H}_i$ denote

$$\mathcal{H}_i \equiv X_i \cap \mathcal{H} = \{v_h \in X_i : q_h^T B_h v_h = 0, \quad \forall q_h \in Y_i\}.$$

Thus, $\mathcal{H}_i$ also sum to $\mathcal{H}$. We divide the solution process into three steps.

1. In the first step we compute a velocity $u_{h,I} \in X_{h,I}$ such that $B_h u_{h,I} = F_h = B_h u_h$. In applications, we usually solve a small coarse mesh problem followed by the concurrent solutions of n subproblems to determine $u_{h,I}$.

2. In the second step, we compute the *discrete divergence free* velocity

$$u_{h,DF} \equiv u_h - u_{h,I} \in \mathcal{H}.$$

Note that, $(u_{h,DF}, p_h)$ satisfies the following equation:

$$\begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_{h,DF} \\ p_h \end{pmatrix} = \begin{pmatrix} W_h - A_h u_{h,I} \\ F_h - B_h u_{h,I} \end{pmatrix} = \begin{pmatrix} W_h - A_h u_{h,I} \\ 0 \end{pmatrix}. \tag{2.13}$$

Thus, by restricting the problem to *discrete divergence free* velocities, we find that if $(u_{h,DF}, p_h), (v_{h,DF}, q_h) \in \mathcal{H} \times Y_h$, then

$$\begin{pmatrix} v_{h,DF} \\ q_h \end{pmatrix}^T \begin{pmatrix} A_h & B_h^T \\ B_h & 0 \end{pmatrix} \begin{pmatrix} u_{h,DF} \\ p_h \end{pmatrix} = \begin{pmatrix} v_{h,DF} \\ q_h \end{pmatrix}^T \begin{pmatrix} W_h - A_h u_{h,I} \\ 0 \end{pmatrix}. \tag{2.14}$$

This leads to an equivalent problem to determine $u_{h,DF}$:

$$\begin{cases} \text{Find } u_{h,DF} \in \mathcal{H}, \text{ such that} \\ v_{h,DF}^T A_h u_{h,DF} = v_{h,DF}^T (W_h - A_h u_{h,I}), \quad \forall v_h \in \mathcal{H}. \end{cases} \tag{2.15}$$

If A_h is positive definite on $\mathcal{H}$, it follows that the problem to find $u_{h,DF}$ is symmetric, positive definite. We use iterative methods, such as the Schwarz methods based on the subspaces $\mathcal{H}_i$ to solve for $u_{h,DF}$ in equation (2.15) Note that, standard iterative methods to solve equation (2.15), may require that the problem be expressed in terms of a basis for $\mathcal{H}$ and that, such a basis may be computationally expensive to construct. However, we can implement the multiplicative and additive versions of the Schwarz alternating method without the use of a basis for the *discrete divergence free* velocity space $\mathcal{H}$. We find the projections onto $\mathcal{H}_i$ by solving a saddle point subproblem on $X_i \times Y_i$. The second step is the most expensive step in the solution process. In most applications involving the multiplicative and additive Schwarz methods, the number of iterations required are independent of the mesh width h. During each iteration, many subproblems can often be solved in parallel.

3. Finally, we compute the pressure p_h. In most applications, the computational cost of this step is the same as or less than the cost of one iteration in step 2. One could use various subproblems to determine pressures in subspaces, and then compute the global pressure p_h. We discuss the specific details later on in this chapter and in the chapter on iterative refinement methods.

In the next section, we apply these methods to the particular case of mixed finite element discretisations of elliptic Neumann problems using the Raviart-Thomas finite element spaces. Recall that the Raviart-Thomas finite element spaces have the property that

$$\textit{discrete divergence free} \Longleftrightarrow L^2\textit{-divergence free.}$$

2.2 Application of the Schwarz methods to mixed finite element discretisations of elliptic Neumann problems using the Raviart-Thomas finite element spaces.

Let Ω be a polygonal region in R^2 triangulated by a *uniformly shape regular* triangulation τ^h which is a refinement of a coarser triangulation τ^H of width H. Thus, every element in τ^H can be expressed as a union of elements in τ^h. The triangulation consists entirely of triangles or entirely of parallelograms. Let the elements in the coarse triangulation τ^H, or as we will refer to them, subdomains, be denoted by $\Omega_1, ..., \Omega_N$. For each subdomain Ω_i, we construct an extension Ω_i^{ext} with $\Omega_i \subset \Omega_i^{ext}$. The extension Ω_i^{ext} is assumed to have $distance(\partial\Omega, \partial\Omega_i^{ext}) \geq cH$, for some positive constant c, independent of i and H. In case $\Omega_i^{ext} \subset \Omega$, we define

$$\Omega_i' \equiv \Omega_i^{ext}.$$

In case $\Omega_i^{ext} \not\subset \Omega$, we cut off the portion of Ω_i^{ext} not in $\overline{\Omega}$, i.e.,

$$\Omega_i' \equiv \Omega_i^{ext} \cap \Omega.$$

See Figure‘2.2. We assume that the extended subdomains Ω_i' are unions of elements in the triangulation τ^h of Ω.

For a fixed non-negative integer r, let $(X_H(\Omega), Y_H(\Omega))$ denote the coarse mesh Raviart-Thomas space of order r for the Neumann problem; i.e.,

$$X_H(\Omega) \equiv V_H(\Omega) \cap \vec{H}_{0,\partial\Omega}(div, \Omega),$$

and

$$Y_H(\Omega) \equiv Q_H(\Omega) \cap L^2(\Omega)/R.$$

Similarly, $(X_h(\Omega), Y_h(\Omega))$ denotes the Raviart-Thomas spaces of order r, based on the fine triangulation τ^h, for the Neumann problem. Let $I_H^h : X_H(\Omega) \to X_h(\Omega)$ denote the interpolation operator for the velocity space. Since $X_H(\Omega) \subset X_h(\Omega)$, I_H^h is the identity on $X_H(\Omega)$. As before, with some abuse of notation, we use $\vec{u}_h$,$\vec{u}_H$, p_h,p_H, etc., to denote functions in the Raviart-Thomas finite element spaces, and interchangeably their representation with respect to the basis used in in the matrix notation. Their use will be clear from the context.

As discussed in the previous section, the solution of the saddle point problem consists of three steps. Step 1 and step 3 are the same for both the multiplicative and additive Schwarz methods. In the case of the multiplicative Schwarz method, the pressure p_h can be determined in step 3 without the use of subproblem solves, and may involve only on the order of N floating point operations. For step 2, we prove that the rates of convergence of both the multiplicative and additive Schwarz methods are independent of h. Numerical results are presented in a later section. They indicate that the rates of convergence for both the multiplicative and additive Schwarz methods are independent of h and H, the fine and coarse mesh parameters, respectively, but depend mildly on c the amount of overlap between subdomains.

2.2.1 Step 1: Reducing the saddle point problem to a symmetric positive definite problem.

To reduce the saddle point problem to a symmetric positive definite problem, we need to compute a discrete velocity $\vec{u}_{h,I} \in X_h(\Omega)$ satisfying:

$$B_h\vec{u}_{h,I} = F_h = B_h\vec{u}_h.$$

First, we solve a coarse-mesh problem to obtain a coarse mesh discrete velocity $\vec{u}_H$ with *normal trace* on $\partial\Omega_j$ compatible with the mean value of $f(x)$ on Ω_j for each j, i.e.,

$$\left\{ \begin{array}{lcll} \text{Find } \vec{u}_H \in X_H(\Omega), \, p_H \in Y_H(\Omega) \text{ such that} & & & \\ a(\vec{u}_H, \vec{v}_H) + b(\vec{v}_H, p_H) & = & 0 & \forall \vec{v}_H \in X_H(\Omega) \\ b(\vec{u}_H, q_H) & = & F(q_H) = \int_\Omega f(x) q_H dx & \forall q_H \in Y_H(\Omega) \end{array} \right. \tag{2.16}$$

Substituting $q_H = \chi_{\Omega_i} \in Y_H(\Omega)$, the characteristic function of Ω_i, into the divergence constraint satisfied by $\vec{u}_H$, we see that:

$$\int_{\Omega_i} f(x)dx = \int_{\Omega_i} \nabla \cdot \vec{u}_H dx = \int_{\partial\Omega_i} \vec{n} \cdot \vec{u}_H ds_x \tag{2.17}$$

i.e., $\vec{u}_H$ has a *normal trace* on $\partial\Omega_i$ which is compatible with the mean value of $f(x)$ on Ω_i for $i = 1, \ldots, N$. Now, $\vec{u}_h - I_H^h \vec{u}_H$ has the property that

$$F_h - B_h I_H^h \vec{u}_H = B^T(\vec{u}_h - I_H^h \vec{u}_H)$$

has mean value zero in each of the subdomains $\Omega_1, \ldots, \Omega_N$. Thus, if we assign zero *normal trace* on each subdomain boundary $\partial\Omega_i$, we would have compatible data to pose N parallel subproblems on $\Omega_1, \ldots, \Omega_N$ for local velocities and pressures $(\vec{\delta u}_h^i, \delta p_h^i)$ using the new right hand side, given by the residual, as follows:

$$\begin{cases} \text{Find } \vec{\delta u}_h^i \in X_h(\Omega_i),\ \delta p_h^i \in Y_h(\Omega_i) \text{ such that} & & \\ a(\vec{\delta u}_h^i, \vec{v}) + b(\vec{v}, \delta p_h^i) = -a(I_H^h \vec{u}_H, v), & \forall \vec{v} \in X_h(\Omega_i) \\ b(\vec{\delta u}_h^i, q) = F(q) - b(I_H^h \vec{u}_H, q), & \forall q \in Y_h(\Omega_i). \end{cases}$$

The local spaces $(X_h(\Omega_i), Y_h(\Omega_i))$ are defined by

$$X_h(\Omega_i) \equiv V_h(\Omega) \cap \vec{H}_{0,\partial\Omega_i}(div, \Omega_i),$$

$$Y_h(\Omega_i) \equiv Q_h(\Omega) \cap L^2(\Omega_i)/R.$$

Since the local problems have compatible boundary conditions, see (2.17), they are well posed. Since the *normal traces* are zero on interface boundaries, they match; thus this composite velocity will be in $X_h(\Omega)$. See Lemma 5 about composite functions. We form the composite velocity by adding up the subdomain velocities $\vec{\delta u}_h^i$.

Let

$$\vec{u}_{h,I} = I_H^h \vec{u}_H + \sum_{i=1}^{M} \vec{\delta u}_h^i$$

By construction, $\vec{u}_{h,I}$ satisfies $B_h \vec{u}_{h,I} = B_h \vec{u}_h = F_h$. Thus $\vec{u}_{h,DF} \equiv \vec{u}_h - \vec{u}_{h,I}$, satisfies:

$$\begin{cases} a(\vec{u}_{h,DF}, \vec{v}) + b(\vec{v}, p_h) = -a(\vec{u}_{h,I}, v), & \forall \vec{v} \in X_h(\Omega) \\ b(\vec{u}_{h,DF}, q) = 0, & \forall q \in Y_h(\Omega), \end{cases} \tag{2.18}$$

i.e., $\vec{u}_{h,DF}$ is *divergence free*. We now show that finding $\vec{u}_{h,DF}$ is a positive definite symmetric problem on the *divergence free* subspace $\mathcal{H}$ defined by:

$$\mathcal{H} \equiv V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div^0, \Omega). \tag{2.19}$$

From equation (2.18), we see that $\vec{u}_{h,DF} \in \mathcal{H}$. Restricting $\vec{v}$ in problem (2.18) to $\mathcal{H}$, we obtain the following equivalent problem:

$$\begin{cases} \text{Find } \vec{u}_{h,DF} \in \mathcal{H} \text{ such that} \\ a(\vec{u}_{h,DF}, \vec{v}) = -a(\vec{u}_{h,I}, \vec{v}) \qquad \forall \vec{v} \in \mathcal{H}. \end{cases} \tag{2.20}$$

Since

$$\mathcal{H} \subset \vec{H}_{0,\partial\Omega}(div^0, \Omega),$$

and since $a(.,.)$ is $\vec{H}(div^0, \Omega)$-elliptic by equation (1.13), it follows that

$$\forall \vec{v}_h \in \mathcal{H}, \qquad a(\vec{v}_h, \vec{v}_h) \geq \alpha \|\vec{v}_h\|^2_{\vec{H}(div,\Omega)}.$$

Therefore, problem (2.20) is a symmetric positive definite problem.

2.2.2 Step 2: Solution of the divergence free symmetric positive definite problem.

We solve problem (2.20) by the use of the multiplicative or additive Schwarz methods. Once $\vec{u}_{h,DF}$ is obtained, the discrete velocity solution $\vec{u}_h$ is determined by letting:

$$\vec{u}_h = \vec{u}_{h,I} + \vec{u}_{h,DF}.$$

First, we introduce some notation. For $i = 1, \ldots, N$, let:

$$\begin{array}{rcl} X_h(\Omega_i') & \equiv & V_h(\Omega) \cap \vec{H}_{0,\partial\Omega_i'}(div, \Omega_i') \\ Y_h(\Omega_i') & \equiv & Q_h \cap L^2(\Omega_i')/R. \end{array} \tag{2.21}$$

These are subspaces of the Raviart-Thomas spaces obtained by restricting them to the extended subdomains $\{\Omega_i'\}$. The functions in these subspaces, when extended by zero outside Ω_i' to the whole domain Ω, lies in the global Raviart-Thomas spaces.

The spaces used in the formulation of the multiplicative and additive Schwarz methods are the *divergence free* subspaces of the Raviart-Thomas velocity spaces. i.e.,

$$\begin{array}{rcl} \mathcal{H} & \equiv & V_h(\Omega) \cap \vec{H}_{0,\partial\Omega}(div^0, \Omega) \\ \mathcal{H}_0 & \equiv & V_H(\Omega) \cap \vec{H}_{0,\partial\Omega}(div^0, \Omega) \\ \mathcal{H}_i & \equiv & V_h(\Omega) \cap \vec{H}_{0,\partial\Omega_i'}(div^0, \Omega_i'). \end{array} \tag{2.22}$$

As stated before, we do not possess a basis for the *divergence free* subspaces $\mathcal{H}, \mathcal{H}_0, \ldots, \mathcal{H}_N$. To implement the Schwarz methods, we need to compute the projections onto the *divergence free* subspaces. For $i = 0, \ldots, N$, let P_i denote the orthogonal projection in the $a(.,.)$ inner product onto $\mathcal{H}_i$. We sometimes use P_H to denote P_0, the orthogonal projection onto the coarse mesh *divergence free* velocity space $\mathcal{H}_0$. To obtain each projection we solve a saddle point subproblem using the

basis for the appropriate Raviart-Thomas subspaces. For instance, to find $P_i\vec{w}$, for $i = 1, \ldots, N$, we solve:

$$\left\{\begin{array}{lll} \text{Find } \vec{w}_h \in X_h(\Omega_i'),\ s_h \in Y_h(\Omega_i') \text{ s.t} & & \\ a(\vec{w}_h, \vec{v}) + b(\vec{v}, s_h) & = a(\vec{w}, \vec{v}) & \forall \vec{v} \in X_h(\Omega_i') \\ b(\vec{w}_h, q) & = 0 & \forall q \in Y_h(\Omega_i') \end{array}\right. \tag{2.23}$$

Thus $P_i\vec{w} = \vec{w}_h \in \mathcal{H}_i$.

For N sufficiently large, let us partition the collection of subdomains $\Omega_1', \ldots, \Omega_N'$ into n colors $1, \ldots, n$, coloring two subdomains the same only if they are disjoint. Let $\mathcal{I}_i$ denote the collection of indices of the subdomains $\{\Omega_j'\}$ of color i, and let C_i be the union of all subdomains $\{\Omega_j'\}$ of color i. Thus,

$$C_i \equiv \cup_{j\in\mathcal{I}_i}\Omega_j'.$$

For all *shape regular* triangulations τ^H, the number of colors n is independent of H and h. One can show this using the fact that the number of neighbouring elements for a *shape regular* triangulation is bounded independently of the mesh widths. For instance, in the case of a rectangular region triangulated by an uniform mesh of rectangles, the number of colors, for suitably chosen overlap among the extended subdomains, is four, independent of H and h. See Figure 2.3, in the section on Numerical results.

Define

$$\mathcal{H}_{C_i} = \sum_{j\in\mathcal{I}_i} \mathcal{H}_j, \tag{2.24}$$

for each color $i = 1, \ldots, n$. This is an orthogonal direct sum. We now state and prove a result concerning the existence of a partition for the *divergence free* Raviart-Thomas subspaces. This result will be used to prove the basic convergence results for the multiplicative and additive Schwarz methods applied to elliptic problems discretised by the Raviart-Thomas mixed finite element method.

Lemma 21 *Let $\mathcal{H}_0, \mathcal{H}_{C_1} \ldots, \mathcal{H}_{C_n}$ be subspaces of $\mathcal{H}$ as defined by equation (2.24). Then, there exists a positive constant μ, independent of h such that for every $\vec{v}_h \in \mathcal{H}$, there exists a partition with $\vec{v}_i \in \mathcal{H}_{C_i}$, for $i = 1, \ldots, n$, and $\vec{v}_0 \in \mathcal{H}_0$ satisfying*

$$\vec{v}_h = \vec{v}_0 + \vec{v}_1 + \cdots + \vec{v}_n,$$

and

$$\sum_{i=0}^{n} a(\vec{v}_i, \vec{v}_i) \le \mu^2 a(\vec{v}_h, \vec{v}_h).$$

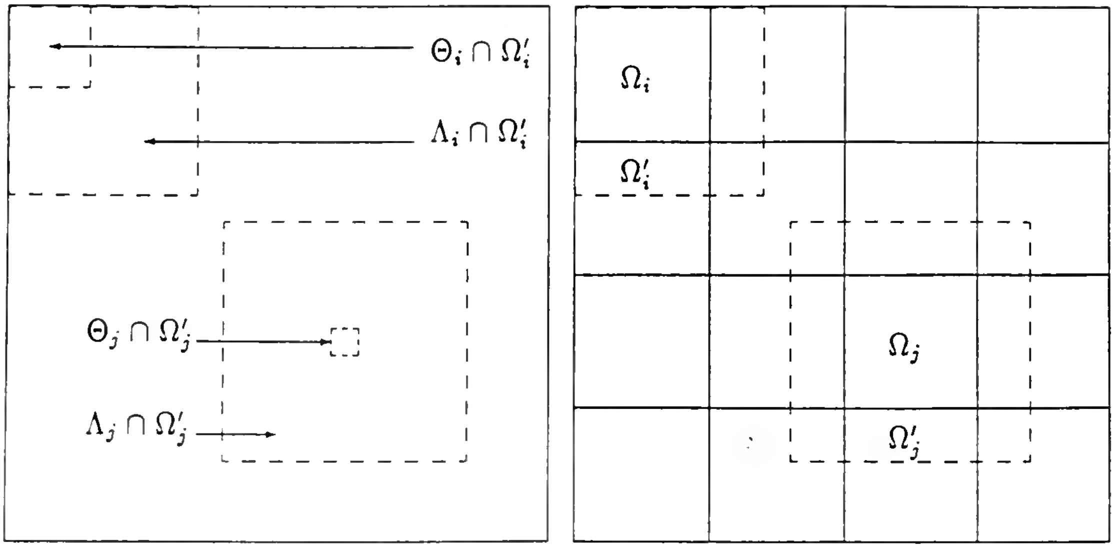


Figure 2.1: Partitioning C_i into Θ_i and Λ_i.

PROOF OF LEMMA. First, we define $\vec{v}_0 = P_0\vec{v}$. Since P_0 is a projection, we obtain:

$$\begin{cases} a(\vec{v}_0, \vec{v}_0) \leq a(\vec{v}, \vec{v}), \\ a(\vec{v} - \vec{v}_0, \vec{v} - \vec{v}_0) \leq a(\vec{v}, \vec{v}). \end{cases} \tag{2.25}$$

Next, we construct a partition for $\vec{v} - \vec{v}_0 = \vec{v}_1 + \ldots + \vec{v}_n$, with each $\vec{v}_i \in \mathcal{H}_{C_i}$ having support in C_i. We construct $\vec{v}_1, \ldots, \vec{v}_n$ sequentially, starting with $\vec{v}_1$. To describe the construction of $\vec{v}_i$ at step i, we partition the colored region C_i into two disjoint sets Θ_i and Λ_i. Though the region C_i may not be connected, it is a union of one or more of the connected subdomains $\{\Omega_j'\}$. The subregions Θ_i and Λ_i will also be the union of connected sets. Θ_i is defined as the subset of C_i which does not intersect the remaining regions $C_{i+1}, \ldots, C_n$. Λ_i is defined as the complement of Θ_i in C_i. Thus, for $i = 1, \ldots, n$:

$$\begin{cases} \Theta_i & \equiv & C_i - (\cup_{j=i+1}^n C_j) = C_i \cap (\cup_{j=i+1}^n C_j)^c \\ \Lambda_i & \equiv & C_i - \Theta_i = C_i \cap \Theta_i^c \end{cases}$$

See Figure 2.1. Note that $\Theta_n = C_n$, $\Lambda_n = \emptyset$, and $\cup_{i=1}^n \Theta_i = \Omega$.

Once $\vec{v}_0, \ldots, \vec{v}_{i-1}$ have been defined on $\cup_{j=1}^{i-1} C_j$, we define $\vec{v}_i$ on Θ_i. Since the remaining regions $\Lambda_i, C_{i+1}, \ldots, C_n$ do not intersect Θ_i, we have no alternative but to define

$$\vec{v}_i \equiv \vec{v} - \sum_{j=0}^{i-1} \vec{v}_j \text{ on } \Theta_i. \tag{2.26}$$

Since the $\vec{v}_j$ are defined to be zero outside C_j, for $j = i+1, \ldots, n$, this gives us

$$\vec{v}_0 + \vec{v}_1 + \ldots + \vec{v}_n = \vec{v} \quad \text{on } \Theta_i.$$

By the definition of $\vec{v}_i$ on Θ_i the norm of $\vec{v}_i$ on Θ_i equals the norm of $\vec{v} - \sum_{j=0}^{i-1} \vec{v}_j$ on Θ_i. Thus,

$$\|\vec{v}_i\|_A \leq \|\vec{v}\|_A + \sum_{j=0}^{i-1} \|\vec{v}_j\|_A.$$

Next, we describe how $\vec{v}_i$ is defined on Λ_i. Since $\vec{v}_i$ is to have support in C_i, the *normal trace* of $\vec{v}_i$ on ∂C_i must be zero. We must also make the *normal trace* of $\vec{v}_i$ on $\partial \Lambda_i$ compatible with the *normal trace* of $\vec{v}_i$ on $\partial \Theta_i$, wherever $\partial \Lambda_i$ and $\partial \Theta_i$ intersect; $\vec{v}_i$ is defined on Θ_i by equation (2.26). Therefore, if we let g_i denote the *normal trace* $\vec{n} \cdot \vec{v}_i$ on $\partial \Lambda_i$, we must have:

$$\begin{cases} g_i = -\gamma_n[\vec{v} - \sum_{j=0}^{i-1} \vec{v}_j] & \text{on } \partial\Lambda_i \cap \partial\Theta_i, \\ g_i = 0 & \text{on } \partial\Lambda_i \cap \partial C_i = \partial\Lambda_i - (\partial\Theta_i \cap \partial\Lambda_i). \end{cases} \tag{2.27}$$

We claim that $\int_{\partial\Lambda_i} g_i ds_x = 0$. There are two cases to consider. To simplify our proof, we consider each connected subdomain Ω'_j that constitutes C_i separately. See Figure 2.1.

1. In case $\Omega_j^{ext} \subset \Omega$, then $\Omega'_j \cap \Lambda_i$ will be ring shaped, and $\partial\Theta_i \cap \Omega'_j$ will be $\partial\Lambda_i \cap \partial\Theta_i \cap \Omega'_j$. Thus, since g_i is the *normal trace* on $\partial\Theta_i \cap \Omega'_j$ of $\vec{v} - \sum_{j=0}^{i-1} \vec{v}_j$, which is *divergence free*, our claim follows.

2. In case $\Omega_j^{ext} \not\subset \Omega$, then $\Omega'_j \cap \Lambda_i$ will be 'L' or 'U' shaped with part of its boundary intersecting $\partial\Omega$. Since $\vec{v} - \sum_{j=0}^{i-1} \vec{v}_j$ has zero *normal trace* on $\partial\Omega$ and since $\vec{v} - \sum_{j=0}^{i-1} \vec{v}_j$ is *divergence free* on $\Theta_i \cap \Omega'_j$, it follows that $\int_{\partial\Lambda_i \cap \partial\Theta_i \cap \Omega'_j} g_i ds_x = 0$.

From this, our claim follows. Furthermore, by an application of the *normal trace* lemma, we can bound

$$\|g_i\|_{H^{-1/2}(\partial\Lambda_i)} \leq c(\Lambda_i)\|\vec{v}_i\|_{\vec{H}(div,\Theta_i)}. \tag{2.28}$$

Using the *normal trace* g_i of $\vec{v}_i$ defined by equation (2.27) on $\partial\Lambda_i$, we define

$$\vec{v}_i|_{\Lambda_i} \equiv E^h g_i \quad \text{on } \Lambda_i, \tag{2.29}$$

where E^h denotes the Raviart-Thomas *divergence free* velocity extension of the *normal trace* g_i. Such an extension exists by the extension theorem for Raviart-Thomas

velocity spaces described in Chapter 1. Furthermore, the extension theorem guarantees that there exists a positive constant $c = c(\Lambda_i)$ with

$$\|\vec{v}_i\|_{\vec{H}(div,\Lambda_i)} = \|E^h g_i\|_{\vec{H}(div,\Lambda_i)} \le c(\Lambda_i)\|g_i\|_{H^{-1/2}(\partial\Lambda_i)}. \tag{2.30}$$

Combining equations (2.26), (2.28), (2.29), (2.30), we obtain, for some positive constant $c(\Lambda_i, \Theta_i)$

$$\|\vec{v}_i\|_{\vec{H}(div,C_i)} \le c(\Lambda_i, \Theta_i)(\sum_{j=0}^{i-1} \|\vec{v}_j\|_{\vec{H}(div,\Omega)}). \tag{2.31}$$

By using equation (2.25), equation (2.31), and induction, it follows that there exists a positive contant c_i, depending on the various regions, and n, which are all independent of h, such that:

$$a(\vec{v}_i, \vec{v}_i) \le c_i^2 a(\vec{v}, \vec{v}).$$

Finally, summing over i, we obtain, for some positive μ independent of h, the following:

$$\sum_{i=0}^{n} a(\vec{v}_i, \vec{v}_i) \le \mu^2 a(\vec{v}, \vec{v}).\square$$

REMARK. The result holds without the use of $\mathcal{H}_0$ and $\vec{v}_0$ in the partition. i.e., given $\vec{v} \in \mathcal{H}$, there exists a partition

$$\vec{v} = \vec{v}_1 + \cdots + \vec{v}_n,$$

with $\vec{v}_i \in \mathcal{H}_{C_i}$, satisfying

$$\sum_{i=1}^{n} a(\vec{v}_i, \vec{v}_i) \le \mu^2 a(\vec{v}, \vec{v}),$$

for some positive constant μ independent of h and $\vec{v} \in \mathcal{H}$. The proof is the same.

Using Lemma 21, we prove the main results about the rate of convergence of the multiplicative and additive Schwarz methods applied to the mixed finite element discretisation of elliptic problems. First, we prove the following result for the *unsymmetrised* multiplicative Schwarz method.

Theorem 3 *For the multiplicative Schwarz method using the subspaces $\mathcal{H}_0, \mathcal{H}_{C_1}, \ldots, \mathcal{H}_{C_n}$ or the subspaces $\mathcal{H}_{C_1}, \ldots, \mathcal{H}_{C_n}$, the convergence factor ρ is bounded less than one, independent of h.*

PROOF. The result follows by use of Lemma 21 and Lemma 18, since the constants involved are inedpendent of h. □

The following result concerns the additive Schwarz method.

Theorem 4 *$P \equiv P_0 + P_1 + \cdots + P_n$ is symmetric and positive definite in the $a(.,.)$ inner product. Furthermore, there exists positive constants δ_1, δ_2, independent of h, such that*

$$\delta_1 \; a(\vec{u}_h, \vec{u}_h) \leq a(P\vec{u}_h, \vec{u}_h) \leq \delta_2 \; a(\vec{u}_h, \vec{u}_h).$$

PROOF OF UPPER BOUND. P is symmetric since it is a sum of orthogonal projections, each of which is symmetric, in the $a(.,.)$ inner product. Using the coloring of the subdomains as before, $C_1, \ldots, C_n$, we may define for each i,

$$P_{C_i} \equiv \sum_{j \in \mathcal{I}_i} P_j.$$

Each P_{C_i} is an orthogonal projection, since it is a sum of orthogonal projections onto subspaces that are mutually orthogonal. Then $P = P_0 + P_{C_1} + \ldots + P_{C_n}$, and so the upper bound δ_2 is $n + 1$. Note that, n, the number of colors for *shape regular* triangulations, is bounded independently of H and h. □

PROOF OF LOWER BOUND. For the lower bound, we use Lemma 15 and Lemma 21 to obtain:

$$\frac{1}{\mu^2} a(\vec{v}, \vec{v}) \leq a(P\vec{v}, \vec{v}),$$

with a constant μ independent of h, but possibly dependent on the geometry of the subdomains.

2.2.3 Step 3: Determining the pressure.

We determine the pressure p_h as follows. Let $(\vec{u}_h, p_h)$ denote the exact solution of the discrete problem (1.14). Then, by construction,

$$\vec{u}_h = \vec{u}_{h,I} + \vec{u}_{h,DF},$$

which was computed in the previous two steps. Recall that, for the mixed formulation of elliptic Neumann problems, $W_h = 0$, thus, p_h satisfies the following equation:

$$\begin{cases} \text{Find } p_h \in Y_h(\Omega) \text{ s.t} & & \\ a(\vec{u}_{h,I} + \vec{u}_{h,DF}, \vec{v}) + b(\vec{v}, p_h) & = 0 & \forall \vec{v} \in X_h(\Omega) \\ b(\vec{u}_{h,I} + \vec{u}_{h,DF}, q) & = 0 & \forall q \in Y_h(\Omega). \end{cases} \tag{2.32}$$

Now, we can find the projection onto the *divergence free* Raviart-Thomas subspaces, for each i, using the new residual. i.e., for $i = 1, \ldots, N$ solve:

$$\left\{ \begin{array}{lcll} \text{Find } \vec{w}_{i,h} \in X_h(\Omega_i'), p_{i,h} \in Y_h(\Omega_i'), \text{ such that} & & & \\ a(\vec{w}_{i,h}, \vec{v}) + b(\vec{v}, p_{i,h}) & = & 0 - a(\vec{u}_{h,I} + \vec{u}_{h,DF}, \vec{v}) = b(\vec{v}, p_h), & \forall \vec{v} \in X_h(\Omega_i') \\ b(\vec{w}_{i,h}, q) & = & 0, & \forall q \in Y_h(\Omega_i') \end{array} \right. \tag{2.33}$$

Each $\vec{w}_{i,h} = 0$, but $p_{i,h}$ will be nonzero in general. Since the *divergence* operator maps $X_h(\Omega_i')$ onto $Y_h(\Omega_i')$, we obtain that

$$\int_{\Omega_i'} (p_{i,h} - p_h) q dx = 0, \quad \forall q \in Y_h(\Omega_i').$$

i.e, $p_{i,h}$ is the L^2 projection of p_h onto $Y_h(\Omega_i')$. By using the uniqueness up to constants of the pressure solutions, we obtain, for $i = 1, \ldots, N$, that :

$$p_{i,h} = p_h + c_i \quad \text{in } \Omega_i',$$

for some constants c_i. Thus, in the intersection between the subdomains, we must have

$$p_{i,h} = p_{j,h} + c_i - c_j \quad \text{in } \Omega_i' \cap \Omega_j'.$$

Thus, once $\{p_{i,h}\}$ are computed, we can add an appropriate constant in each extended subdomain Ω_i' for each $i = 1, \ldots, N$, sequentially to get a consistent pressure in $\cup_{j=1}^{i} \Omega_j'$. In the last step we obtain a global pressure which differs from the individual subdomain pressures $\{p_{i,h}\}$ by some constant in each subdomain Ω_i'. When this global pressure is normalised to have mean value zero in Ω, it equals the exact pressure solution p_h defined by equation (2.12).

2.3 Numerical results.

In this section, we present the results of numerical tests with the multiplicative and additive Schwarz algorithms for mixed finite element discretisations of elliptic problems. We choose the domain of the elliptic problem to be the unit square $[0,1] \times [0,1]$ in the $x - y$ plane and let the coefficients $\mathcal{A}(x,y)$ of the elliptic operator be diagonal and piecewise constant:

$$\mathcal{A}(x,y) = a(x,y) I,$$

where I denotes the identity matrix and $a(x,y)$ is a piecewise constant function with a jump across $x = \frac{1}{2}$:

$$a(x,y) = \begin{cases} 1 & 0 \le x < 0.5 \\ J & 0.5 \le x \le 1 \end{cases} \tag{2.34}$$

The ratio of the coefficients, J, varies between 1 and 10^{-6}. To obtain the discretisation, we use the lowest order Raviart-Thomas spaces defined on a rectangular mesh. We note that for a uniform mesh, such as in Figure 2.3, we can represent the Raviart-Thomas spaces, as a tensor product of two spaces, one along each axis, cf. Glowinski and Wheeler [16].

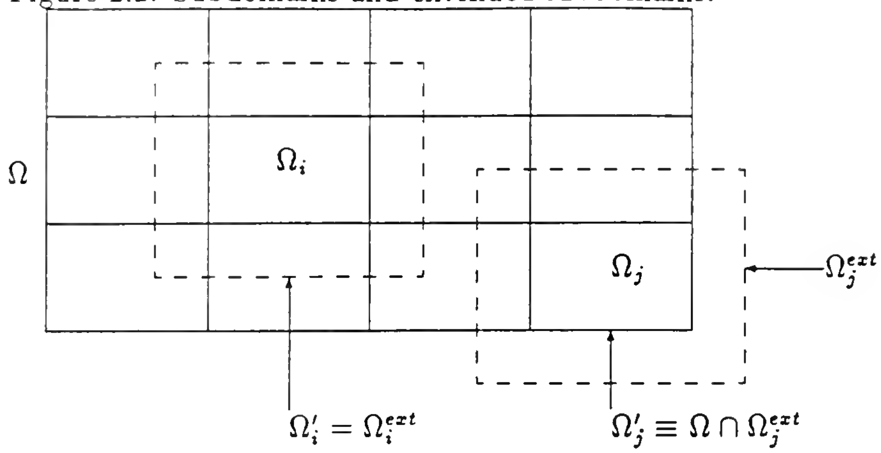
For a positive integer n, the fine mesh on the unit square is taken to be the $n * n$ uniform square mesh. Let $h = 1/n$ denote the fine mesh parameter, then the diameter of the square element is $\sqrt{2}h$. The coarse mesh is obtained by dividing the square into $n_s * n_s$ square elements, called subdomains. We assume that n_s divides n. Thus, each subdomain (coarse mesh element) Ω_i contains $(n/n_s) * (n/n_s)$ fine mesh elements. We let the coarse mesh parameter be $H = 1/n_s$; thus the diameter of the coarse mesh element is $\sqrt{2}/n_s$. To obtain an overlap between the subdomains, we introduce an integer parameter n_o. The extended subdomains are obtained by centering a square Ω_i^{ext} of $(n_o + n/n_s + n_o) * (n_o + n/n_s + n_o)$ elements at the center of subdomain Ω_i, and removing the part outside the unit square. The resulting subregions are denoted by $\Omega_i' \equiv \Omega_i^{ext} \cap \Omega$. See Figure 2.2, for an example without the elements in the fine mesh drawn in. Thus, in the case of the interior subdomains, Ω_i' contains $(n/n_s + 2n_o)^2$ elements. The results are presented for various choices of n, n_s, n_o and for various choices of discontinuities in the coefficients of the elliptic operator.

We make a few remarks about the linear system obtained by using the lowest order Raviart-Thomas finite element spaces on a uniform rectangular mesh. We recall that the linear system has the form

$$\begin{bmatrix} A_h & B_h^T \\ B_h & 0 \end{bmatrix} \begin{bmatrix} u_h \\ p_h \end{bmatrix} = \begin{bmatrix} W_h \\ F_h \end{bmatrix}, \tag{2.35}$$

where A_h is a symmetric, positive definite matrix. Since the velocity unknowns are the values of the *normal component* on the edges of the rectangular elements, one on each edge, we have as many velocity unknowns as there are element edges in the interior of the unit square. Thus, there are $2n(n-1)$ such degrees of freedom. For diagonal $\mathcal{A}(x,y)$, the velocity unknowns on the vertical edges are uncoupled from the

Figure 2.2: Subdomains and extended subdomains.



velocity unknowns on the horizontal edges in the matrix A_h; however, in the stiffness matrix, there will be couplings via the pressure. For a suitable ordering of the velocity unknowns, the leading block A_h of the stiffness matrix is tridiagonal, with couplings between adjacent vertical edges and adjacent horizontal edges.

For the Laplacian, i.e., $\mathcal{A}(x,y) = I$, one can verify that A_h is well conditioned, uniformly in h. However, if $\mathcal{A}(x,y)$ contains a discontinuity, then the condition number of A_h will be proportional to the magnitude of the jump, uniformly in h, and for large discontinuities, the matrix A_h can be ill conditioned.

If a basis for the pressure space $Q_h(\Omega)$ is used in the construction of the stiffness matrix, instead of $Y_h(\Omega) = Q_h(\Omega)/R$, then the stiffness matrix has a null space corresponding to a constant pressure. Thus, in that case, B_h^T has a null space of dimension 1 corresponding to a constant pressure. Recall that for the lowest order Raviart-Thomas space on a rectangular mesh, there is one pressure node associated with each rectangular element. Thus, the dimension of $Q_h(\Omega)$ is n^2. We note that the block matrix B_h does not contain any information about the coefficients of the elliptic operator, $\mathcal{A}(x,y)$, since it is an approximation of the *divergence* map.

The Schur complement associated with p_h is $B_h A_h^{-1} B_h^T$, and it is ill conditioned as $h \longrightarrow 0$, even for $\mathcal{A}(x,y) = I$. This is because this Schur complement represents a discretisation of the elliptic problem in p. We can show that the Schur complement is ill-conditioned by using the *inf-sup* condition and the fact that the $a(.,.)$ norm is

not equivalent to the $\vec{H}(div, \Omega)$ norm, except for *divergence free* functions.

Next, we discuss a preconditioned iterative method that we use to solve the local problems. Let

$$\begin{bmatrix} \tilde{A}_h & \tilde{B}_h^T \\ \tilde{B}_h & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_h \\ \tilde{p}_h \end{bmatrix} = \begin{bmatrix} \tilde{W}_h \\ \tilde{F}_h \end{bmatrix} \tag{2.36}$$

denote a local stiffness matrix. Note that, the extended subdomains in the interior contain $2(n/n_s + 2n_o)(n/n_s + 2n_o - 1)$ velocity unknowns, and $(n/n_s + 2n_o)^2$ pressure unknowns. To solve the local saddle point problems, we use the following Schur complement method:

1. Solve $\tilde{A}_h u_0 = \tilde{W}_h$, in the first step. For uniform meshes, $\tilde{A}_h$ is tridiagonal for a suitable ordering of the unknowns, provided that $\mathcal{A}(x,y)$ is isotropic, which is the case when $\mathcal{A}(x,y)$ is diagonal. Thus, we use direct solvers in this step.
2. Solve $\tilde{B}_h \tilde{A}_h^{-1} \tilde{B}_h^T \tilde{p}_h = \tilde{F}_h - \tilde{B}_h u_0$, using a conjugate gradient method. As mentioned earlier, even for $\mathcal{A}(x,y) = I$, this matrix becomes progressively ill conditioned as $h \longrightarrow 0$. In the case that $\mathcal{A}(x,y)$ is piecewise constant, the condition number of $\tilde{B}_h \tilde{A}_h^{-1} \tilde{B}_h^T$ is at least on the order of the jump. We have found that for extreme values of the jump, the conjugate gradient method does not always converge, even if the number of iterations is larger than the size of the system. We therefore use a diagonally scaled conjugate gradient algorithm, using the diagonal of the matrix as a preconditioner. We remark that other preconditioners can also be chosen, such as a finite difference matrix associated with the discretisation of the elliptic problem for p. See Wheeler and Gonzalez [33].
3. In the third and final step, we compute the solution of

$$\tilde{A}_h u_1 = -\tilde{B}_h^T p_0.$$

The solution equals $\tilde{u} = u_0 + u_1$ and $\tilde{p}$.

Since the mesh is uniform, the interpolation map from the coarse to the fine mesh is simple. As mentioned in earlier sections, it is possible to color the extended subdomains so that the iteration in the multiplicative Schwarz method is more parallelisable. In one of the algorithms we tested, we colored the entire collection of extended subdomains using four colors. In addition we have the coarse mesh problem. Figure 2.3 describes the four colors. We refer to the resulting algorithm as the four color-five subspace multiplicative Schwarz algorithm. Surprisingly, as the results

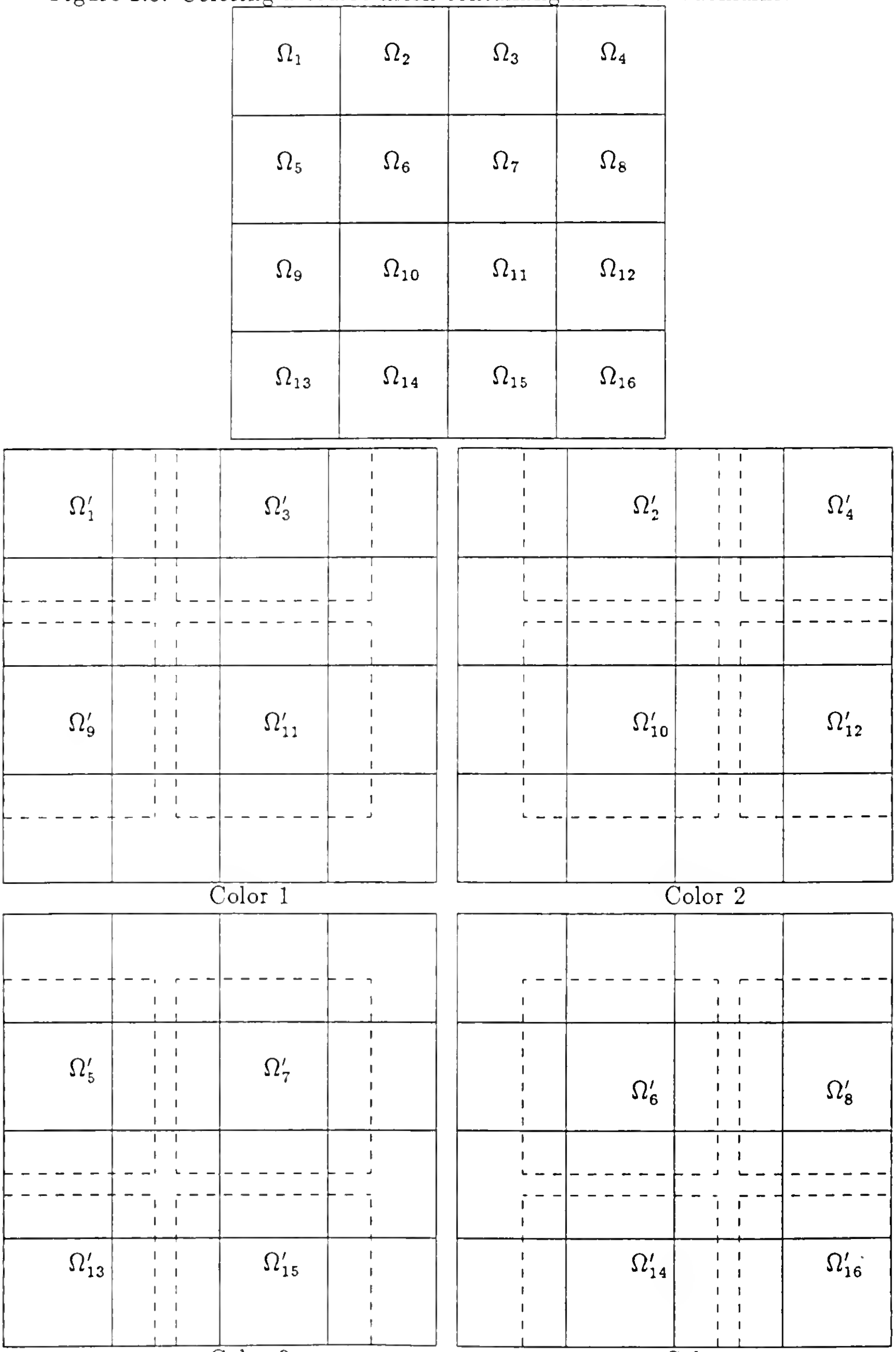
indicate, this method converges faster than a multiplicative Schwarz method based on the standard lexicographic ordering of the extended subdomains.

We remark that in testing the algorithms, we found that the inclusion of the coarse model problem enhanced the convergence, and provided a rate of convergence quite independent of the mesh parameters n, n_s and n_o. Without this coarse model, which is quite an inexpensive addition, the rate of convergence deteriorated with an increasing number of subdomains. This is to be expected for the following reason. Using the properties of the Green's function, we see that the solution of an elliptic problem with $f(x)$ having support in $\Omega^* \subset \Omega$, can have values significantly different from zero in all of Ω. Thus, in a domain decomposition method in which there is only a transfer of information between adjacent subdomains during each iteration, we would be able to devise a problem for which at least n_D iterations are required for convergence. Here, n_D denotes the largest number of steps required to transfer information between any two subdomains in Ω; see Widlund [37]. The coarse model problem is an attempt to include a mechanism to transfer information to all the subdomains, during each iteration, i.e., a mechanism for a "global transportation of information".

For the multiplicative Schwarz methods, we store the local pressures that are obtained in computing the projections onto the local *divergence free* spaces, for the most recent iteration. However, we do not use these local pressures to compute the residuals. With these stored local pressures, we are able to compute the global pressure p_h once the iteration is stopped, in $O(n^2)$ flops. Since the Schwarz method involves incrementing the current iterate by adding an update obtained by solving a subproblem, the maginitude of the error in the local problem solver contrains the accuracy of the Schwarz method. In other words, if the local problems are accurate only to within a tolerance of ϵ, the iterates in the Schwarz method would, in general, be accurate to only the same tolerance. However, this need not be the case in the acclerated *symmetrised* multiplicative Schwarz method, which we did not test.

In testing the algorithms, we found that for problems in which the jump ratio $1/J$ is large, the relative error of the computed pressure solution was larger than the relative error of the computed velocity solution by a factor on the order of $1/J$. We observed this phenomenon, even when we used direct methods to solve the saddle point problems. We attribute this to round-off error.

Figure 2.3: Coloring a coarse mesh containing sixteen subdomains.



We remark that for the additive Schwarz method, we compute the right hand side

$$g_h = \sum_{j=0}^{n_s^2} P_j \vec{u}_{h,DF},$$

by adding the projections onto the local *divergence free* spaces, see section 2.1.2 and section 2.2.2. Thus, g_h is *divergence free,* and the iterates in the conjugate gradient algorithm remain in the *divergence free* space $\mathcal{H}$, within the error of the subproblem solvers.

We now present the numerical results in table 2.1, table 2.2 and table 2.3. We list the convergence factor for the various mesh parameters and jumps. The jumps listed are the ratios $1/J$; see equation (2.34). The convergence factor is the average factor by which the relative error is reduced, during an iteration, using the standard euclidean mesh norm. They are virtually the same in the A norm and in the maximum norm. The exact solutions were randomly chosen using a random number generator using the uniform distribution on $(-2, 2)$. The number of velocity unknowns for $n = 16$ is 480 and the number of pressure unknowns is 256. For $n = 24$, there are 1104 velocity unknowns and 576 pressure unknowns. For $n = 32$, there are 1984 velocity unknowns and 1024 pressure unknowns. For $n = 40$, there are 3120 velocity unknowns and 1600 pressure unknowns. For $n = 60$, there are 7080 velocity unknowns and 3600 pressure unknowns.

Conclusions. The numerical results indicate that the convergence factor of the four color-five subspace multiplicative Schwarz method is about the square of the convergence factor of the additive Schwarz method with a coarse model. Thus, the additive Schwarz method with the coarse model would require about twice as many as iterations as for the four color-five subspace multiplicative Schwarz method, to reduce the relative error by the same factor. The results also indicate that the lexicographic multiplicative Schwarz algorithm with a coarse model is slower than the four color-five subspace multiplicative Schwarz method. For all of these methods, the results indicate that the rate of convergence is quite independent of the jump J, and mildly dependent on the amount of overlap ratio, $n_o n_s/n$. In addition, the results indicate a rate of convergence that is overall, quite independent of the mesh parameters, for fixed overlap ratios.

Recall that our proofs on the rate of convergence, did not make significant use of the coarse model. Without this, it is not possible to prove that the rate of convergence

Table 2.1: Results for the additive Schwarz method with a coarse model.

$h^{-1} = n$	Decompositions $n_s * n_s$	n_o	Overlap ratio $n_o n_s / n$	Jump J^{-1}	Convergence factor
16	2 * 2	1	1/8	1	0.3428
16	2* 2	1	1/8	10^6	0.4287
16	2 * 2	2	1/4	1	0.2645
16	2* 2	2	1/4	10^6	0.3804
16	4 * 4	1	1/4	1	0.3434
16	4* 4	1	1/4	10^6	0.4057
16	4 * 4	2	1/2	1	0.3443
16	4* 4	2	1/2	10^6	0.3693
16	8 * 8	1	1/2	1	0.3417
16	8* 8	1	1/2	10^6	0.3494
24	4* 4	1	1/6	1	0.3788
24	4* 4	1	1/6	10^6	0.4106
24	4* 4	2	1/3	1	0.3399
24	4* 4	2	1/3	10^6	0.3405
24	8* 8	1	1/3	1	0.3483
24	8* 8	1	1/3	10^6	0.3589
32	4* 4	1	1/8	1	0.4227
32	4* 4	1	1/8	10^6	0.4436
32	4* 4	2	1/4	1	0.3335
32	4* 4	2	1/4	10^6	0.3730
32	8* 8	1	1/4	1	0.3528
32	8* 8	1	1/4	10^6	0.3694
32	8* 8	2	1/2	1	0.3492
32	8* 8	2	1/2	10^6	0.3520
40	4* 4	1	1/10	1	0.4402
40	4* 4	1	1/10	10^6	0.4843
40	4* 4	2	1/5	1	0.3597
40	4* 4	2	1/5	10^6	0.3852
40	8* 8	1	1/5	1	0.3721
40	8* 8	1	1/5	10^6	0.3935
40	8* 8	2	2/5	1	0.3325
40	8* 8	2	2/5	10^6	0.3436
40	10 * 10	1	1/4	1	0.3552
40	10 * 10	1	1/4	10^6	0.3796
40	10 * 10	2	1/2	1	0.3530
40	10 * 10	2	1/2	10^6	0.3590
40	20 * 20	1	1/2	1	0.3455

Table 2.2: Results for the four color-five subspace multiplicative Schwarz method.

$h^{-1} = n$	Decompositions $n_s * n_s$	n_o	Overlap ratio $n_o n_s / n$	Jump J^{-1}	Convergence factor
16	$2 * 2$	1	1/8	1	0.1652
16	$2 * 2$	1	1/8	10^6	0.1627
16	$2 * 2$	2	1/4	1	0.0548
16	$2 * 2$	2	1/4	10^6	0.0538
16	$4 * 4$	1	1/4	1	0.0596
16	$4 * 4$	1	1/4	10^6	0.0612
16	$4 * 4$	2	1/2	1	0.0276
16	$4 * 4$	2	1/2	10^6	0.0335
16	$8 * 8$	1	1/2	1	0.0451
16	$8 * 8$	1	1/2	10^6	0.0543
24	$4 * 4$	1	1/6	1	0.0939
24	$4 * 4$	1	1/6	10^6	0.0987
24	$4 * 4$	2	1/3	1	0.0500
24	$4 * 4$	2	1/3	10^6	0.0514
24	$8 * 8$	1	1/3	1	0.0530
24	$8 * 8$	1	1/3	10^6	0.0535
32	$4 * 4$	1	1/8	1	0.1584
32	$4 * 4$	1	1/8	10^6	0.1629
32	$4 * 4$	2	1/4	1	0.0576
32	$4 * 4$	2	1/4	10^6	0.0576
32	$8 * 8$	1	1/4	1	0.0622
32	$8 * 8$	1	1/4	10^6	0.0621
32	$8 * 8$	2	1/2	1	0.0360
32	$8 * 8$	2	1/2	10^6	0.0402
40	$4 * 4$	1	1/10	1	0.2535
40	$4 * 4$	1	1/10	10^6	0.2552
40	$4 * 4$	2	1/5	1	0.0723
40	$4 * 4$	2	1/5	10^6	0.0688
40	$8 * 8$	1	1/5	1	0.0748
40	$8 * 8$	1	1/5	10^6	0.0748
40	$8 * 8$	2	2/5	1	0.0406
40	$8 * 8$	2	2/5	10^6	0.0391
40	$10 * 10$	1	1/4	1	0.0644
40	$10 * 10$	1	1/4	10^6	0.0641
40	$10 * 10$	2	1/2	1	0.0360
40	$10 * 10$	2	1/2	10^6	0.0392
40	$20 * 20$	1	1/2	1	0.0748
60	$10 * 10$	1	1/6	10^6	0.0884
60	$10 * 10$	2	1/3	1	0.0683

Table 2.3: Results for the lexicographic multiplicative Schwarz method.

$h^{-1} = n$	Decompositions $n_s * n_s$	n_o	Overlap ratio $n_o n_s / n$	Jump J^{-1}	Convergence factor
16	2 * 2	1	1/8	1	0.1613
16	2* 2	1	1/8	10^6	0.1466
16	2 * 2	2	1/4	1	0.0558
16	2* 2	2	1/4	10^6	0.0486
16	4 * 4	1	1/4	1	0.0836
16	4* 4	1	1/4	10^6	0.0888
16	4 * 4	2	1/2	1	0.0545
16	4* 4	2	1/2	10^6	0.0631
16	8 * 8	1	1/2	1	0.1143
16	8* 8	1	1/2	10^6	0.1253
24	4* 4	1	1/6	1	0.0926
24	4* 4	1	1/6	10^6	0.1035
24	4* 4	2	1/3	1	0.0635
24	4* 4	2	1/3	10^6	0.0772
24	8* 8	1	1/3	1	0.1048
24	8* 8	1	1/3	10^6	0.1236
32	4* 4	1	1/8	1	0.1550
32	4* 4	1	1/8	10^6	0.1727
32	4* 4	2	1/4	1	0.0773
32	4* 4	2	1/4	10^6	0.0911
32	8* 8	1	1/4	1	0.1027
32	8* 8	1	1/4	10^6	0.1170
32	8* 8	2	1/2	1	0.1132
32	8* 8	2	1/2	10^6	0.1245
40	4* 4	1	1/10	1	0.2499
40	4* 4	1	1/10	10^6	0.2644
40	4* 4	2	1/5	1	0.0735
40	4* 4	2	1/5	10^6	0.0985
40	8* 8	1	1/5	1	0.0944
40	8* 8	1	1/5	10^6	0.1013
40	8* 8	2	2/5	1	0.0987
40	8* 8	2	2/5	10^6	0.1137
40	10 * 10	1	1/4	1	0.1011
40	10 * 10	1	1/4	10^6	0.1120
40	10 * 10	2	1/2	1	0.1159
40	10 * 10	2	1/2	10^6	0.1299

is independent of the coarse mesh parameter $H = 1/n_s$, since without a mechanism for a "global transportation of information" during each iteration, the rate of convergence of the algorithms will depend on H. We have, so far, been unable to prove that the rate of convergence is independent of H. We mention that, for standard finite element discretisations of elliptic problems, Dryja and Widlund [13] have proved that the rate of convergence of the additive Schwarz method is independent of both H and h, in both two and three dimensions.

Chapter 3

Iterative refinement methods for Raviart-Thomas elements.

In this Chapter, we discuss iterative methods to solve the discrete linear systems arising from the discretisation of mixed formulations of elliptic problems using the Raviart-Thomas spaces on a repeatedly refined mesh. These iterative methods are referred to as the FAC and AFAC algorithms, in a similar context for the discretisation of standard elliptic problems. First, we describe the iteratively refined mesh, and the Raviart-Thomas spaces. Following that we describe the FAC and AFAC algorithms and present some theoretical bounds for their rates of convergence. We also include some quantitative bounds for the rate of convergence of the FAC algorithm, in the special case of discretisation using the lowest order Raviart-Thomas spaces on triangles.

3.1 Definition of the refined meshes and spaces.

Let $\Omega_0 \equiv \Omega \subset R^2$ be a polygonal domain triangulated by a *uniformly shape regular* mesh of width h_0, denoted by $\tau^{h_0}(\Omega_0)$. Let $\Omega_1 \subset \Omega_0$ be a polygonal subregion of the triangulation where we wish to refine the mesh. If there is neccesity for still further refinement, say, another $l-1$ levels of refinement, we introduce a sequence of nested polygonal regions where we intend to refine the mesh, repeatedly:

$$\Omega_0 \supset \Omega_1 \supset \cdots \supset \Omega_{l-1} \supset \Omega_l.$$

Before we define the repeatedly refined mesh, we introduce a standard multilevel grid defined on Ω_0. We consider a sequence of refined *uniformly shape regular* meshes, of

mesh widths,

$$h_l < h_{l-1} < \cdots < h_1 < h_0,$$

with the corresponding partitions

$$\tau^{h_0}(\Omega_0) \subset \tau^{h_1}(\Omega_0) \subset \cdots \subset \tau^{h_{l-1}}(\Omega_0) \subset \tau^{h_l}(\Omega_0). \tag{3.1}$$

We assume that the vertices of the elements in the triangulations (3.1) never lie on the interior of an edge of another element. By a refinement we mean the following: Each element K of a coarser mesh is divide into two or more smaller elements, such that the vertices of the smaller elements do not lie on the interior of an edge of a neighbouring element in the refined mesh. In a standard case of refinement, as in the case of many multigrid methods, each element in the coarser mesh is divided into four subelements by connecting the midpoints of the coarse mesh element. In this case, the mesh width of the refined mesh is reduced by a factor of 2.

For $i = 1, \ldots, l$, the *uniformly shape regular* mesh $\tau^{h_i}(\Omega_0)$, of width $h_i < h_{i-1}$, is assumed to be constructed by refinement of the *uniformly shape regular* mesh $\tau^{h_{i-1}}(\Omega_0)$. We assume that Ω_i, for $i = 1, \ldots, l$, can be expressed as a union of elements in $\tau^{h_{i-1}}(\Omega_0)$. To obtain the *iteratively refined* mesh, we consider restrictions of each mesh $\tau^{h_i}(\Omega_0)$ to the subregion Ω_i to obtain $\tau^{h_i}(\Omega_i)$.

DEFINITION. We define the *repeatedly refined* mesh $\tau^{h_0,\ldots,h_l}(\Omega)$ as consisting of those elements which are in any of the various refined meshes $\tau^{h_i}(\Omega_0)$ restricted to $\Omega_i - \Omega_{i+1}$, where we define $\Omega_{l+1} \equiv \emptyset$. i.e.,

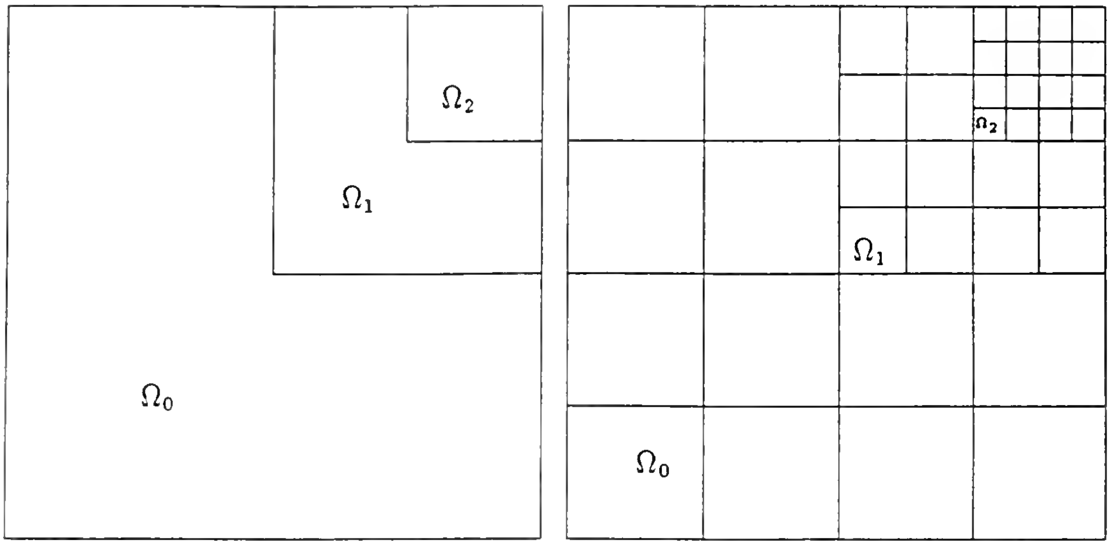
$$\tau^{h_0,\ldots,h_l}(\Omega) \equiv \cup_{i=0}^{l} \tau^{h_i}(\Omega_i - \Omega_{i+1}).$$

REMARK. In the case of the *repeatedly refined* mesh, the vertices of elements in the refined regions such as along $\partial\Omega_i$, may lie on the interior of edges of coarser elements in neighbouring regions. See Figure 3.1. Also, the regions, Ω_i need not be connected, and this can lead to increased parallelism in the numerical algorithms to be studied.

DEFINITION. We define the refined mesh Raviart-Thomas spaces:

$$X_{h_0,\ldots,h_l}(\Omega) \equiv X_{h_0}(\Omega_0) + X_{h_1}(\Omega_1) + \cdots + X_{h_{l-1}}(\Omega_{l-1}) + X_{h_l}(\Omega_l),$$

Figure 3.1: A repeatedly refined rectangular mesh with 2 levels of refinement.



where $X_{h_i}(\Omega_i)$ are the Raviart-Thomas velocity spaces with zero *normal trace* on $\partial\Omega_i$, as used in the discretisation of the Neumann problem on Ω_i. Of course,

$$X_{h_0,\dots,h_l}(\Omega) \subset \vec{H}_{0,\partial\Omega}(div,\Omega).$$

Similarly, we define

$$Y_{h_0,\dots,h_l}(\Omega) \equiv Y_{h_0}(\Omega_0) + Y_{h_1}(\Omega_1) + \cdots + Y_{h_{l-1}}(\Omega_{l-1}) + Y_{h_l}(\Omega_l),$$

where $Y_{h_i}(\Omega_i)$ are the Raviart-Thomas pressure spaces with zero *mean value* on Ω_i as used in the discretisation of the Neumann problem on Ω_i. Thus $Y_{h_0,\dots,h_l}(\Omega)$ is also a subset of $L^2(\Omega)/R$.

We use the refined mesh Raviart-Thomas spaces $(X_{h_0,\dots,h_l}(\Omega), Y_{h_0,\dots,h_l}(\Omega))$, defined on the repeatedly refined mesh, to obtain our discretisation of saddle point problem (1.7). Thus, our discrete problem is:

$$\left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } u_h \in X_{h_0,\dots,h_l}(\Omega),\ p_h \in Y_{h_0,\dots,h_l}(\Omega) \text{ such that}} \\ a(\vec{u}_h,\vec{v}) + b(\vec{v},p_h) & = & 0, & \forall \vec{v} \in X_{h_0,\dots,h_l}(\Omega) \\ b(\vec{u}_h,q) & = & \int_\Omega f(x)q(x)dx, & \forall q \in Y_{h_0,\dots,h_l}(\Omega), \end{array} \right. \tag{3.2}$$

where $a(.,.)$ and $b(.,.)$ are defined by equation (1.12). For well posedness, we need to check the following two conditions.

1. First, we need to check that $a(.,.)$ is *positive definite* on

$$X_{h_0,\ldots,h_l;0}(\Omega) \equiv \{\vec{u}_h \in X_{h_0,\ldots,h_l}(\Omega) : b(\vec{u}_h, q) = 0, \quad \forall q \in Y_{h_0,\ldots,h_l}(\Omega)\},$$

equipped with the $\vec{H}(div,\Omega)$ norm, with a constant independent of h. We will show that

$$\text{if } \vec{u}_h \in X_{h_0,\ldots,h_l;0}(\Omega) \text{ then } \nabla \cdot \vec{u}_h = 0,$$

i.e., *discrete divergence free* implies *divergence free* in the $L^2(\Omega)$ sense. Then, $a(.,.)$ will be positive definite on $X_{h_0,\ldots,h_l;0}(\Omega)$ with constant α since

$$X_{h_0,\ldots,h_l;0}(\Omega) \subset \vec{H}_{0,\partial\Omega}(div^0, \Omega),$$

and since

$$a(\vec{v}, \vec{v}) \geq \alpha \|\vec{v}\|^2_{\vec{H}(div,\Omega)}, \quad \forall \vec{v} \in \vec{H}(div^0, \Omega),$$

see Chapter 1, equation (1.13).

So consider, $\vec{u}_h \in X_{h_0,\ldots,h_l}(\Omega)$. Then, by definition,

$$\vec{u}_h = \vec{u}_0 + \cdots + \vec{u}_l, \quad \text{where} \quad \vec{u}_i \in X_{h_i}(\Omega_i), \forall i,$$

and

$$\nabla \cdot \vec{u}_h = \nabla \cdot \vec{u}_0 + \cdots + \nabla \cdot \vec{u}_l, \quad \text{where} \quad \nabla \cdot \vec{u}_i \in Y_{h_i}(\Omega_i), \forall i.$$

By definition of $Y_{h_0,\ldots,h_l}(\Omega)$ as the sum of the Raviart-Thomas pressure spaces defined on the refined meshes, it follows that $\nabla \cdot \vec{u}_h \in Y_{h_0,\ldots,h_l}(\Omega)$. This implies that

$$\text{if } b(\vec{u}_h, q) = 0, \ \forall q \in Y_{h_0,\ldots,h_l}(\Omega) \text{ then } \int_\Omega |\nabla \cdot \vec{u}_h|^2 dx = 0.$$

Thus *discrete divergence free* implies *divergence free* in the L^2 sense.

2. Next, we need to check that $b(.,.)$ satisfies a *uniform inf-sup* condition on the refined mesh Raviart-Thomas spaces $(X_{h_0,\ldots,h_l}(\Omega), Y_{h_0,\ldots,h_l}(\Omega))$. Given $q_h \in Y_{h_0,\ldots,h_l}(\Omega)$, we will show that there exists a $\vec{u}_h \in X_{h_0,\ldots,h_l}(\Omega)$, satisfying $\nabla \cdot \vec{u}_h = q_h$, and

$$\|\vec{u}_h\|_{\vec{H}(div,\Omega)} \leq c(\Omega_0, \ldots, \Omega_l)\|q_h\|_{L^2(\Omega)}, \tag{3.3}$$

for some positive constant $c(\Omega_0, \ldots, \Omega_l)$ independent of $h_0, \ldots, h_l$. From this it follows that

$$\sup_{\vec{w}_h \in X_{h_0,\ldots,h_l}(\Omega)-\{0\}} \frac{b(\vec{w}_h, q_h)}{\|\vec{w}_h\|_{\vec{H}(div,\Omega)}\|q_h\|_{L^2(\Omega)}} \geq \frac{b(\vec{u}_h, q_h)}{\|\vec{u}_h\|_{\vec{H}(div,\Omega)}\|q_h\|_{L^2(\Omega)}}$$

$$= \frac{\int_\Omega (\nabla \cdot \vec{u}_h) q_h dx}{\|\vec{u}_h\|_{\vec{H}(div,\Omega)} \|q_h\|_{L^2(\Omega)}} = \frac{\|q_h\|^2_{L^2(\Omega)}}{\|\vec{u}_h\|_{\vec{H}(div,\Omega)} \|q_h\|_{L^2(\Omega)}} \geq \frac{1}{c(\Omega_0, \ldots, \Omega_l)},$$

which is the *uniform inf-sup* condition with constant $1/c(\Omega_0, \ldots, \Omega_l)$.

Now we prove condition (3.3). Let P_{i,L^2} denote the L^2 projection onto $Y_{h_i}(\Omega_i)$, for $i = 0, \ldots, l$. We will find an $L^2(\Omega)$ orthogonal decomposition of

$$q_h = q_0 + \cdots + q_l,$$

with $q_i \in Y_{h_i}(\Omega_i)$. For $i = 0$, we let

$$q_0 \equiv P_{0,L^2} q_h,$$

and for $i = 1, \ldots, l$, we let

$$q_i \equiv P_{i,L^2} P^{\perp}_{i-1,L^2} \cdots P^{\perp}_{0,L^2} q_h.$$

Note that, for $1 \geq i$,

$$q_h - \sum_{j=0}^{i-1} q_j = (I - P_{0,L^2} - \sum_{j=1}^{i-1} P_{j,L^2} P^{\perp}_{j-1,L^2} \cdots P^{\perp}_{0,L^2}) q_h$$

$$= P^{\perp}_{i-1,L^2} \cdots P^{\perp}_{0,L^2} q_h,$$

and thus,

$$q_i = P_{i,L^2}(q_h - \sum_{j=0}^{i-1} q_j).$$

Since both q_h and q_0 have mean value zero, it follows that q_0 is the L^2 projection onto $Y_{h_0}(\Omega_0) \ominus \mathcal{P}_0(\Omega_0)$. By discontinuity of the pressure spaces across element boundaries, and the fact that $Y_{h_0}(\Omega_0 - \Omega_1) = Y_{h_0,\ldots,h_l}(\Omega_0 - \Omega_1)$, it follows that

$$q_h - q_0 = 0, \quad \text{on } \Omega_0 - \Omega_1.$$

Since, the characteristic function of Ω_1, denoted by χ_{Ω_1}, is in $Y_{h_0}(\Omega_0) \oplus \mathcal{P}_0(\Omega_0)$, it follows that

$$q_h - q_0 \text{ has mean value zero on } \Omega_1.$$

Similarly, for $i = 2, \ldots, l$, we obtain by induction, that

$$q_h - \sum_{j=0}^{i-1} q_j = 0 \quad \text{on } \Omega_0 - \Omega_i,$$

and

$$q_h - \sum_{j=0}^{i-1} q_j \in Y_{h_i}(\Omega_i) + \cdots + Y_{h_l}(\Omega_l).$$

Thus q_i, which is the L^2 projection of $q_h - \sum_{j=0}^{i-1} q_j$ onto $Y_{h_i}(\Omega_i)$, will satisfy

$$q_h - \sum_{j=0}^{i} q_j = 0 \qquad \text{on } \Omega_0 - \Omega_{i+1},$$

by using the discontinuity of the discrete pressures across element boundaries, and that

$$Y_{h_i}(\Omega_i - \Omega_{i+1}) = Y_{h_0,\ldots,h_l}(\Omega_i - \Omega_{i+1}).$$

Note that,

$$q_l = P_{l,L^2}(q_h - \sum_{j=0}^{l-1} q_j) = q_h - \sum_{j=0}^{l-1} q_j.$$

Thus,

$$q_h = q_0 + \cdots + q_l.$$

We show that $\{q_i\}$ are mutually orthogonal as follows. First, for $i = 1$, we will show that $q_0 \perp q_1$. By construction,

$$q_h - q_0 \in Y_{h_0}(\Omega_0)^{\perp}, \tag{3.4}$$

and thus $q_h - q_0$ has support in Ω_1. Since q_1 is the L^2 projection of $q_h - q_0$ onto $Y_{h_1}(\Omega_1)$, it follows that

$$\int_{\Omega}(q_h - q_0 - q - 1)q_0 dx = \int_{\Omega_1}(q_h - q_0 - q - 1)q_0 dx = 0, \tag{3.5}$$

because $q_0|_{\Omega_1} \in Y_{h_1}(\Omega_1)$. By using equation (3.4) and equation (3.5), we obtain that

$$\int_{\Omega_1} q_0 q_1 dx = 0.$$

Next, we show that if $1 \leq i$, then q_{i+1} is orthogonal to $q_0, \ldots, q_i$. By definition, q_{i+1} is the L^2 projection of $q_h - \sum_{j=0}^{i} q_j$, which has support in Ω_{i+1}, onto $Y_{h_{i+1}}(\Omega_{i+1})$. Thus,

$$\int_{\Omega_0}(q_h - \sum_{j=0}^{i+1} q_j)q_k dx = \int_{\Omega_{i+1}}(q_h - \sum_{j=0}^{i+1} q_j)q_k dx = 0,$$

for $k \leq i$, since $q_k|_{\Omega_{i+1}} \in Y_{h_j}(\Omega_j)$, for $j \geq k$. We decompose

$$\int_{\Omega_{i+1}}(q_h - \sum_{j=0}^{i+1} q_j)q_k dx = \int_{\Omega_{i+1}}(q_h - \sum_{j=0}^{i} q_j)q_k dx + \int_{\Omega_{i+1}}(-q_{i+1})q_k dx.$$

The first part is zero by the definition of q_i, since $q_k|_{\Omega_{i+1}} \in Y_{h_i}(\Omega_i)$. Therefore, the second part satisfies

$$0 = \int_{\Omega_{i+1}} (-q_{i+1}) q_k dx,$$

which shows that $q_k \perp q_{i+1}$ for $k \leq i$. Thus,

$$q_h = q_0 + \cdots + q_l, \qquad \text{where } q_i \in Y_{h_i}(\Omega_i),$$

is an orthogonal decomposition of q_h.

Since the triangulation $\tau^{h_i}(\Omega_i)$ is *uniformly shape regular* on Ω_i, we know that the *uniform inf-sup* condition holds on the local Raviart-Thomas spaces for the Neumann problem, $(X_{h_i}(\Omega_i), Y_{h_i}(\Omega_i))$. Thus, there exists a positive constant, $c_i(\Omega_i)$ such that given $q_i \in Y_{h_i}(\Omega_i)$, there exists $\vec{v}_i \in X_{h_i}(\Omega_i)$ such that $\nabla \cdot \vec{v}_i = q_i$ and

$$||\vec{v}_i||_{\vec{H}(div,\Omega_i)} \leq c_i(\Omega_i)||q_i||_{L^2(\Omega_i)}.$$

By defining

$$\vec{v}_h \equiv \vec{v}_0 + \cdots + \vec{v}_l,$$

we obtain,

$$\nabla \cdot \vec{v}_h = q_o + \cdots + q_l = q_h,$$

and

$$\begin{aligned} ||\vec{v}_h||_{\vec{H}(div,\Omega)} &\leq \sum_{i=0}^{l} ||\vec{v}_i||_{\vec{H}(div,\Omega_i)} \leq \sum_{i=0}^{l} c_i(\Omega_i)||q_i||_{L^2(\Omega_i)} \\ &\leq \max(c_0, \ldots, c_l)\sqrt{l+1}||q_h||_{L^2(\Omega)}. \end{aligned}$$

Note that, the constants $c_0, \ldots, c_l$ are independent of $h_0, \ldots, h_l$. Therefore, the repeatedly refined Raviart-Thomas spaces satisfy an *inf-sup* condition, with a constant independent of $h_0, \ldots, h_l$, and possibly depending on l, the number of levels of refinement, and the geometry of the refined regions.

Since the repeatedly refined, composite, spaces lead to stable discretisations, the error in discretisation is given by

$$\begin{aligned} &||\vec{u} - \vec{u}_h||_{\vec{H}(div,\Omega)} + ||p - p_h||_{L^2(\Omega)} \\ &\leq \inf_{(\vec{v}_h, q_h) \in X_{h_0,\ldots,h_l}(\Omega) \times Y_{h_0,\ldots,h_l}(\Omega)} \{||\vec{u} - \vec{v}_h||_{\vec{H}(div,\Omega)} + ||p - q_h||_{L^2(\Omega)}\}, \end{aligned}$$

where $(\vec{u}, p)$ is the solution of the continuous problem and $(\vec{u}_h, p_h)$ is the solution of the discrete problem. We assume that our choice of refinement leads to a small error. Throughout the rest of this Chapter, we consider iterative methods to solve saddle point linear system (3.2).

3.2 FAC and AFAC methods for Raviart-Thomas elements.

The fast adaptive composite grid methods (FAC), and the asynchronous fast adaptive composite grid methods (AFAC) were originally introduced as iterative methods to solve positive definite linear systems arising from the standard discretisations of elliptic problems using repeatedly refined meshes; cf. McCormick and Thomas [26], Mandel and McCormick [24], Widlund [36] and Dryja and Widlund [14]. The two level case is discussed in Mandel and McCormick [24] and also Bramble, Ewing, Pasciak and Schatz [6]. We refer to Widlund [36] and Dryja and Widlund [14] for the analysis of the many level case. The FAC and AFAC algorithms are actually the multiplictive and additive versions of the Schwarz method, respectively, applied to special subspaces.

As discussed in chapter 2, our application to the saddle point problem is accomplished in three steps. The second step is where the Schwarz iterative methods are applied, and this step is, computationally, the most expensive step. Steps 1 and 3, are virtually the same for the FAC and AFAC methods, although less work is involved in the third step of the FAC method. The steps are:

1. Find $\vec{u}_{h,I} \in X_{h_0,\dots,h_l}(\Omega)$ such that

$$B_h \vec{u}_{h,I} = F_H.$$

2. Find $\vec{u}_{h,DF} \in \mathcal{H}_{h_0,\dots,h_l}$ such that

$$a(\vec{u}_{h,DF}, \vec{v}) = a(-\vec{u}_{h,I}, \vec{v}), \qquad \forall \vec{v} \in \mathcal{H}_{h_0,\dots,h_l},$$

   where $\mathcal{H}_{0,\dots,l} \equiv X_{h_0,\dots,h_l}(\Omega) \cap \vec{H}(div^0, \Omega)$.

3. Determine the pressure p_h.

3.2.1 Step 1: Reduction to a divergence free problem.

To find a discrete velocity $\vec{u}_{h,I}$ with the same *divergence* as the discrete velocity solution $\vec{u}_h$, we sequentially solve problems on each of the nested regions determining a *normal trace* compatible with $f(x)$ on the next subregion. First, we solve the

following problem on the coarsest mesh $\tau^{h_0}(\Omega_0)$:

$$\begin{cases} \text{Find } \vec{u}_{h_0,I} \in X_{h_0}(\Omega_0),\ p_{h_0,I} \in Y_{h_0}(\Omega_0) \text{ such that} \\ a(\vec{u}_{h_0,I},\vec{v}) + b(\vec{v},p_{h_0,I}) = 0 \qquad \forall \vec{v} \in X_{h_0}(\Omega_0) \\ b(\vec{u}_{h_0,I},q) = \int_\Omega f(x)q\,dx \qquad \forall q \in Y_{h_0}(\Omega_0) \end{cases} \tag{3.6}$$

Substituting $q \in Y_{h_0}(\Omega_0)$ with support in $\Omega_0 - \Omega_1$, we see that the solution of equation (3.6), $\vec{u}_{h_0,I}$ satisfies

$$\nabla \cdot \vec{u}_{h_0,I} = \nabla \cdot \vec{u}_h, \qquad \text{on } \Omega_0 - \Omega_1,$$

where $\vec{u}_h$ is the discrete velocity solution, problem (3.2) (repeatedly refined saddle point problem). Note that, on Ω_1, the *divergence* of $\vec{u}_h - \vec{u}_{h_0,I}$ will not be zero in general. However, $\vec{u}_{h_0,I}$ will have *normal trace* on $\partial\Omega_1$ compatible with the *divergence* constraint. i.e.,

$$\int_{\Omega_1} \nabla \cdot \vec{u}_{h_0,I}dx = \int_{\Omega_1} f(x)dx = \int_{\partial\Omega_1} \gamma_n \vec{u}_{h_0,I}ds_x.$$

Thus, $f(x) - \nabla \cdot \vec{u}_{h_0,I}$ has mean value zero on Ω_1, and so zero *normal trace* on $\partial\Omega_1$ is compatible with it. Thus, we could pose a problem on Ω_1 using the new residual, and repeat the procedure, sequentially, on $\Omega_1, \ldots, \Omega_l$. We define

$$\vec{u}_{h,I} = \sum_{i=0}^{l} \vec{u}_{h_i,I},$$

where each $\vec{u}_{h_i,I}$ is obtained by solving the following local problem, for $i = 1, \ldots, l$:

$$\begin{cases} \text{Find } \vec{u}_{h_i,I} \in X_{h_i}(\Omega_i),\ p_{h_i,I} \in Y_{h_i}(\Omega_i) \text{ such that} \\ a(\vec{u}_{h_i,I},\vec{v}) + b(\vec{v},p_{h_i,I}) = 0 \qquad \forall \vec{v} \in X_{h_i}(\Omega_i) \\ b(\vec{u}_{h_i,I},q) = \int_\Omega [f(x) - \sum_{j=0}^{i-1} \nabla \cdot \vec{u}_{h_j,I}]q\,dx \qquad \forall q \in Y_{h_i}(\Omega_i) \end{cases} \tag{3.7}$$

Note that, by construction, for $i = 0, \ldots, l-1$,

$$\int_{\Omega_i} \chi_{\Omega_{i+1}}(\nabla \cdot \vec{u}_{h_i,I})dx = \int_{\Omega_i} \chi_{\Omega_{i+1}}[f(x) - \sum_{j=0}^{i-1} \nabla \cdot \vec{u}_{h_j,I}]dx$$

$$= \int_{\partial\Omega_{i+1}} \gamma_n \vec{u}_{h_i,I}ds_x.$$

Thus, $f(x) - \sum_{j=0}^{i} \nabla \cdot \vec{u}_{h_j,I}$ has mean value zero on Ω_{i+1}, which makes problem (3.7) for $i+1$ well posed. Note, that, for $i = 0, \ldots, l$,

$$\nabla \cdot \vec{u}_{h_i,I} = \nabla \cdot \vec{u}_h - \sum_{j=0}^{i-1} \nabla \cdot \vec{u}_{h_j,I}, \qquad \text{on } \Omega_i - \Omega_{i+1},$$

where, $\Omega_{l+1} \equiv \emptyset$. Thus, we obtain

$$\nabla \cdot \vec{u}_h = \nabla \cdot \vec{u}_{h,I},$$

i.e., $B_h \vec{u}_{h,I} = F_h$. The equation satisfied by $(\vec{u}_h - \vec{u}_{h,I}, p_h)$ is:

$$\begin{cases} a(\vec{u}_{h,DF}, \vec{v}) + b(\vec{v}, p_h) & = & 0 - a(\vec{u}_{h,I}, \vec{v}) & \forall \vec{v} \in X_{h_0,\ldots,h_l}(\Omega) \\ b(\vec{u}_{h,DF}, q) & = & 0, & \forall q \in Y_{h_0,\ldots,h_l}(\Omega), \end{cases} \tag{3.8}$$

where $\vec{u}_{h,DF} \equiv \vec{u}_h - \vec{u}_{h,I}$. Thus, $\vec{u}_{h,DF}$ is *divergence free* and is the solution of the following equivalent problem:

$$\begin{cases} \text{Find } \vec{u}_{h,DF} \in \mathcal{H}_{h_0,\ldots,h_l} \text{ such that} \\ a(\vec{u}_{h,DF}, \vec{v}) = -a(\vec{u}_{h,I}, \vec{v}) \qquad \forall \vec{v} \in \mathcal{H}_{h_0,\ldots,h_l}, \end{cases} \tag{3.9}$$

where $\mathcal{H}_{h_0,\ldots,h_l} \equiv X_{h_0,\ldots,h_l}(\Omega) \cap \vec{H}(div^0, \Omega)$.

3.2.2 Step 2: Solution of the divergence free symmetric positive definite problem.

We solve problem (3.9) by the use of the multiplicative or additive Schwarz methods, or as referred to in this context, the FAC and AFAC algorithms, respectively. Once $\vec{u}_{h,DF}$ is obtained, the discrete velocity solution $\vec{u}_h$ is determined by letting

$$\vec{u}_h = \vec{u}_{h,I} + \vec{u}_{h,DF}.$$

First, we introduce some notation. For $i = 0, \ldots, l$, let

$$\mathcal{H}_{h_i} \equiv X_{h_i}(\Omega_i) \cap \vec{H}_{0,\partial\Omega_i}(div^0, \Omega_i).$$

These are *divergence free* subspaces of the Raviart-Thomas velocity spaces defined on the refined meshes, $\tau^{h_0}(\Omega_0), \ldots, \tau^{h_l}(\Omega_l)$. The functions in these subspaces, when extended by zero outside Ω_i to the whole domain Ω, lie in the global space $\mathcal{H}_{h_0,\ldots,h_l}$. These are the subspaces used in the formulation of the FAC and AFAC methods.

Basis for the *divergence free* subspaces $\mathcal{H}_{h_0,\ldots,h_l}, \mathcal{H}_{h_0}, \ldots, \mathcal{H}_{h_l}$ are not available. To implement the Schwarz methods, we need to compute the projections onto the *divergence free* subspaces. We compute these by solving saddle point subproblems. For $i = 0, \ldots, l$, let P_i denote the orthogonal projection in the $a(.,.)$ inner product onto $\mathcal{H}_{h_i}$. To obtain each projection, we solve a saddle point subproblem using the

basis for the appropriate local refined mesh Raviart-Thomas spaces. For instance, to find $P_i\vec{w}$, for $i = 0, \ldots, l$, we solve:

$$\begin{cases} \text{Find } \vec{w}_{h_i} \in X_{h_i}(\Omega_i),\ s_{h_i} \in Y_{h_i}(\Omega_i) \text{ such that} & & \\ a(\vec{w}_{h_i}, \vec{v}) + b(\vec{v}, s_{h_i}) = a(\vec{w}, v) & \forall \vec{v} \in X_{h_i}(\Omega_i) \\ b(\vec{w}_{h_i}, q) = 0 & \forall q \in Y_{h_i}(\Omega_i) \end{cases} \tag{3.10}$$

Thus $P_i\vec{w} = \vec{w}_{h_i} \in \mathcal{H}_{h_i}$.

We now state and prove a result concerning the existence of a partition for the *divergence free* Raviart-Thomas subspaces. This result will be used to prove convergence results for the FAC and AFAC methods applied to elliptic problems discretised by the Raviart-Thomas mixed finite element method.

Lemma 22 *Let $\mathcal{H}_{h_0}, \ldots, \mathcal{H}_{h_l}$ be* divergence free *subspaces of $\mathcal{H}_{h_0,\ldots,h_l}$, defined on the refined meshes, as described before. Then, there exists a positive constant μ, independent of $h_0, \ldots, h_l$ such that for every $\vec{v}_h \in \mathcal{H}_{h_0,\ldots,h_l}$, there exists a partition with $\vec{v}_i \in \mathcal{H}_{h_i}$, for $i = 0, \ldots, l$, satisfying*

$$\vec{v}_h = \vec{v}_0 + \vec{v}_1 + \cdots + \vec{v}_l,$$

and

$$\sum_{i=0}^{l} a(\vec{v}_i, \vec{v}_i) \leq \mu^2 a(\vec{v}_h, \vec{v}_h).$$

PROOF OF LEMMA. Given $\vec{u}_h \in \mathcal{H}_{h_0,\ldots,h_l}$, we define $\vec{u}_i$ for $i = 0, \ldots, l-1$ by:

$$\vec{u}_i = \begin{cases} \vec{u}_h - \sum_{j=0}^{i-1} \vec{u}_j & \text{on } \Omega_i - \Omega_{i+1} \\ E^{h_i} g_i & \text{on } \Omega_{i+1} \end{cases}$$

where $g_i \equiv \gamma_n(\vec{u}_h - \sum_{j=0}^{i-1} \vec{u}_j)$ on $\partial\Omega_{i+1}$, and E^{h_i} denotes the Raviart-Thomas *divergence free* velocity extension onto $V_{h_i}(\Omega_{i+1}) \cap \vec{H}(div^0, \Omega_{i+1})$, as given by the extension theorem in chapter 1. We define $\vec{u}_l$ as:

$$\vec{u}_l \equiv \vec{u}_h - \sum_{j=0}^{l-1} \vec{u}_j, \qquad \text{on } \Omega_l.$$

Note that, by the *normal trace* lemma

$$\|g_i\|_{H^{-1/2}(\partial\Omega_{i+1})} \leq c_i \|\vec{u}_h - \sum_{j=0}^{i-1} \vec{u}_j\|_{\vec{H}(div,\Omega_{i+1})},$$

and by the extension theorem

$$\|E^{h_i} g_i\|_{\vec{H}(div,\Omega_{i+1})} \leq c_i \|g_i\|_{H^{-1/2}(\partial\Omega_{i+1})}.$$

Combining these, we obtain,

$$a(\vec{u}_i, \vec{u}_i) \leq c_i^2 a(\vec{u}_h - \sum_{j=0}^{i-1} \vec{u}_j, \vec{u}_h - \sum_{j=0}^{i-1} \vec{u}_j),$$

for some positive constant c_i independent of the mesh parameter h_i.

For $i = 0$, we obtain

$$a(\vec{u}_0, \vec{u}_0) \leq c_0^2 a(\vec{u}_h, \vec{u}_h),$$

for some positive constant c_0 independent of h_0. By induction, we obtain, for $i = 1, \ldots, l$:

$$a(\vec{u}_i, \vec{u}_i) \leq (1 + c_0^2)(1 + 2c_1^2) \cdots (1 + ic_{i-1}^2)(i + 1)c_i^2 a_{\Omega_i}(\vec{u}_h, \vec{u}_h).$$

Summing over all i, we obtain an estimate for μ^2:

$$\mu^2 \leq [\{1 + (i + 1)c_l^2\}\{1 + lc_{l-1}^2\} \cdots \{1 + 2c_1^2\}\{1 + c_0^2\} - 1]. \quad \square$$

REMARK. The preceeding result implies that for a fixed number of levels l, the constant μ is independent of the mesh parameters $h_0, \ldots, h_l$. Since we used the extension theorem to obtain a bound for μ, it could possibly depend on the subdomain geometry. However, we will present a result later on that gives a bound for the rate of convergence of the FAC method which is independent of the geometry of the subdomains Ω_i.

We consider applications of this lemma.

Theorem 5 *The error propagation map of the FAC algorithm, i.e., the multiplicative Schwarz method using the subspaces $\mathcal{H}_{h_0}, \mathcal{H}_{h_1}, \ldots, \mathcal{H}_{h_l}$ of $\mathcal{H}_{h_0,\ldots,h_l}$, has a convergence factor ρ which is independent of $h_0, \ldots, h_l$.*

PROOF. We use Lemma 19 and Lemma 22 to obtain the result. Later on we present another proof of this result. $\square$

The following result concerns a variant of the AFAC algorithm, cf. Dryja and Widlund [14] and Widlund [36]. The standard version of the AFAC algorithm involves the use of projections P_{i+1}^i, which are projections onto the $\mathcal{H}_{i+1} \cap \mathcal{H}_i$. However, we consider only a variant of that algorithm, which involves P_i, the projections onto $\mathcal{H}_i$.

Theorem 6 *Let P_i denote the $a(.,.)$ orthogonal projection onto $\mathcal{H}_{h_i}$. Then,*

$$P \equiv P_0 + P_1 + \cdots + P_l,$$

is symmetric and positive definite in the $a(.,.)$ inner product. Furthermore, there exists positive constants δ_1, δ_2, independent of $h_0, \ldots, h_l$, such that

$$\delta_1 \; a(\vec{u}_h, \vec{u}_h) \leq a(P\vec{u}_h, \vec{u}_h) \leq \delta_2 \; a(\vec{u}_h, \vec{u}_h).$$

PROOF OF UPPER BOUND. P is symmetric since it is a sum of orthogonal projections, each of which is symmetric, in the $a(.,.)$ inner product. An upper bound for δ_2 is $l+1$, since the norm of each projection is bounded by one. We can show that this upper bound cannot be improved as follows. Let $\vec{u}_{h_0} \in \mathcal{H}_{h_l} \cap \mathcal{H}_{h_0}$, which will be nonempty as h_0 is made smaller. Then, $P_i\vec{u}_{h_0} = \vec{u}_{h_0}$, for each i, and so the upper bound is $l+1$. $\square$

PROOF OF LOWER BOUND. To obtain the lower bound, we use Lemma 15, (Lions' lemma), and Lemma 22 to obtain a constant μ independent of $h_0, \ldots, h_l$ such that:

$$\frac{1}{\mu^2} a(\vec{v}, \vec{v}) \leq a(P\vec{v}, \vec{v}). \qquad \square$$

Next, we prove a lemma estimating the condition number of the *symmetrised* FAC method. Following that we give an application using the *strengthened Cauchy inequality* to give a bound for the condition number which is independent of the geometry of the subregions Ω_i, and the mesh parameters $h_0, \ldots, h_l$. See Dryja and Widlund [36], and Mandel and McCormick [24].

DEFINITION. By the *symmetrised* FAC algorithm, we mean the multiplicative Schwarz method applied to solve equation (3.9), using defect correction on the subspaces

$$\mathcal{H}_{h_l}, \mathcal{H}_{h_{l-1}}, \cdots, \mathcal{H}_{h_1}, \mathcal{H}_{h_0}, \mathcal{H}_{h_1}, \cdots, \mathcal{H}_{h_{l-1}}, \mathcal{H}_{h_l},$$

in that order.

In the following, we describe the steps in more detail and introduce some notation. Let $\tilde{a}(\vec{u}, \vec{v})$ denote the bilinear form associated with the *symmetrised* FAC preconditioner, and let $f(.)$ denote a linear functional. Then, the solution $\tilde{\vec{u}}$ of

$$\tilde{a}(\tilde{\vec{u}}, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_0,\ldots,h_l},$$

is given by

$$\tilde{\vec{u}} \equiv \vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1 + \vec{u}_0 + \vec{u}_1^* + \cdots + \vec{u}_{l-1}^* + \vec{u}_l^*, \tag{3.11}$$

where

- $\vec{u}_l \in \mathcal{H}_{h_l}$ is the solution of

$$a(\vec{u}_l, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_l};$$

- $\vec{u}_{l-1} \in \mathcal{H}_{h_{l-1}}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1}, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_{l-1}};$$

$$\vdots$$

- $\vec{u}_1 \in \mathcal{H}_{h_1}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_1};$$

- $\vec{u}_0 \in \mathcal{H}_{h_0}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1 + \vec{u}_0, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_0};$$

- $\vec{u}_1^* \in \mathcal{H}_{h_1}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1 + \vec{u}_0 + \vec{u}_1^*, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_1};$$

$$\vdots$$

- $\vec{u}_{l-1}^* \in \mathcal{H}_{h_{l-1}}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1 + \vec{u}_0 + \vec{u}_1^* + \cdots + \vec{u}_{l-1}^*, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_{l-1}};$$

  and

- $\vec{u}_l^* \in \mathcal{H}_{h_l}$ is the solution of

$$a(\vec{u}_l + \vec{u}_{l-1} + \cdots + \vec{u}_1 + \vec{u}_0 + \vec{u}_1^* + \cdots + \vec{u}_{l-1}^* + \vec{u}_l^*, \vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_l}.$$

The following Lemma describes another representation for the bilinear form $\tilde{a}(\vec{u}, \vec{v})$ associated with the *symmetrised* FAC algorithm. See also Dryja and Widlund [36] and Mandel and McCormick [24].

Lemma 23 *The bilinear form $\tilde{a}(\vec{u},\vec{v})$ has the following representation:*

$$\tilde{a}(\vec{u},\vec{v}) = \sum_{i=0}^{l} M_i(\vec{u},\vec{v}), \tag{3.12}$$

where

$$M_l(\vec{u},\vec{v}) \equiv a(P_l\vec{u}, P_l\vec{v})$$
$$M_i(\vec{u},\vec{v}) \equiv a(P_i H^i_{\Omega_{i+1}}\vec{u}, P_i H^i_{\Omega_{i+1}}\vec{v}) \quad \text{for } i = 0,\ldots,l-1,$$

P_i denotes the $a(.,.)$ orthogonal projection onto $\mathcal{H}_{h_i}$, and

$H^i_{\Omega_{i+1}}\vec{u}$ is defined for $\vec{u} \in \mathcal{H}_{h_0,\ldots,h_l}$ and $0 \le i \le l-1$ by:

$$\begin{cases} H^i_{\Omega_{i+1}}\vec{u} \in \mathcal{H}_{h_0} + \cdots + \mathcal{H}_{h_i}, & \\ H^i_{\Omega_{i+1}}\vec{u}(x) \equiv \vec{u}(x), & \text{for } x \in \Omega_0 - \Omega_{i+1}, \\ a(H^i_{\Omega_{i+1}}\vec{u},\vec{v}) = 0, & \forall \vec{v} \in \mathcal{H}_{h_i} \cap \mathcal{H}_{h_{i+1}}, \end{cases}$$

i.e., $H^i_{\Omega_{i+1}}\vec{u}$ is the same as $\vec{u}$ everywhere except on Ω_{i+1} where it is the harmonic extension of $\vec{u}$, in the sense of $\mathcal{H}_{h_i} \cap \mathcal{H}_{h_{i+1}}$, with the same normal trace *as $\vec{u}(x)$ on $\partial\Omega_{i+1}$.*

PROOF. Let $\bar{\vec{u}}$ be the solution of

$$\tilde{a}(\bar{\vec{u}},\vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_0,\ldots,h_l},$$

i.e., $\bar{\vec{u}}$ is given by equation (3.11). We will show that

$$M_i(\bar{\vec{u}},\vec{v}) = \begin{cases} a(\vec{u}_i,\vec{v}) & \forall \vec{v} \in \mathcal{H}_{h_0} + \cdots + \mathcal{H}_{h_i} \\ 0 & \forall \vec{v} \in \mathcal{H}_{h_{i+1}} + \cdots + \mathcal{H}_{h_l} \end{cases} \tag{3.13}$$

From this follows that

$$\tilde{a}(\bar{\vec{u}},\vec{v}) = \sum_{j=i}^{l} a(\vec{u}_j,\vec{v}) = a(\vec{u}_l + \cdots + \vec{u}_i,\vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_i}.$$

Since this holds for each i and since

$$\mathcal{H}_{h_0,\ldots,h_l} = \mathcal{H}_{h_0} + \cdots + \mathcal{H}_{h_l},$$

follows that

$$\tilde{a}(\bar{\vec{u}},\vec{v}) = f(\vec{v}), \quad \forall \vec{v} \in \mathcal{H}_{h_0,\ldots,h_l}.$$

i.e., $\tilde{a}(\vec{u},\vec{v})$ is represented by equation (3.12).

We first prove equation (3.13), first, for l, then for the remaining i. Let $\vec{v} \in \mathcal{H}_{h_0,\ldots,h_l}$. Then

$$a(P_l\bar{\vec{u}}, P_l\vec{v}) = a(P_l\vec{u}_l, P_l\vec{v}) + a(P_l(\vec{u}_{l-1} + \cdots + \vec{u}_l), P_l\vec{v}).$$

Using the definition of $\{\vec{u}_i\}$, and the properties of orthogonal projections, we obtain

$$a(P_l\vec{\tilde{u}}, P_l\vec{v}) = a(\vec{u}_l, \vec{v}) + a(\vec{u}_{l-1} + \cdots + \vec{u}_l^*, P_l\vec{v}).$$

The second term is zero since $\vec{u}_{l-1} + \cdots + \vec{u}_l^*$ is orthogonal to $\mathcal{H}_{h_l}$. Thus,

$$a(P_l\vec{\tilde{u}}, P_l\vec{v}) = a(\vec{u}_l, \vec{v}).$$

Next, we consider the case of $i = 0, \ldots, l-1$. There are two parts to prove.

1. Let

   $$\vec{v} \in \mathcal{H}_{h_{i+1}} + \cdots + \mathcal{H}_{h_l}.$$

   Then, since the support of $\vec{v}$ is in Ω_{i+1}, using the definition gives $H^j_{\Omega_{j+1}}\vec{v} = 0$ for $j \leq i$. Thus, $M_j(\vec{u}, \vec{v}) = 0$, for $j \leq i$.

2. Let

   $$\vec{v} \in \mathcal{H}_{h_0} + \cdots + \mathcal{H}_{h_i}.$$

   We split $\vec{\tilde{u}}$ in four parts,

   $$\vec{\tilde{u}} = (\vec{u}_l + \cdots + \vec{u}_{i+1}) + (\vec{u}_i) + (\vec{u}_{i-1} + \cdots + \vec{u}_i^*) + (\vec{u}_{i+1}^* + \cdots + \vec{u}_l^*).$$

   Since the first and fourth parts have support in Ω_{i+1}, the application of $H^i_{\Omega_{i+1}}$ to them gives zero. Using the definition of $\vec{u}_l, \ldots, \vec{u}_i$, we see that

   $$a(\vec{u}_i, \vec{v}) = 0, \quad \forall \vec{v} \in \mathcal{H}_{h_i} \cap \mathcal{H}_{h_{i+1}}.$$

   Hence, $H^i_{\Omega_{i+1}}\vec{u}_i = \vec{u}_i$ and $P_i H^i_{\Omega_{i+1}}\vec{u}_i = \vec{u}_i$. Similarly, we obtain that

   $$a(\vec{u}_{i-1} + \cdots + \vec{u}_i^*, \vec{v}) = 0, \quad \forall \vec{v} \in \mathcal{H}_{h_i},$$

   and

   $$H^i_{\Omega_{i+1}}(\vec{u}_{i-1} + \cdots + \vec{u}_i^*) = (\vec{u}_{i-1} + \cdots + \vec{u}_i^*).$$

   So far, we have obtained

   $$M_i(\vec{\tilde{u}}, \vec{v}) = a(\vec{u}_i, P_i H^i_{\Omega_{i+1}}\vec{v}) + a(P_i(\vec{u}_{i-1} + \cdots + \vec{u}_i^*), P_i H^i_{\Omega_{i+1}}\vec{v}).$$

   But, the second term is zero since $P_i H^i_{\Omega_{i+1}}\vec{v} \in \mathcal{H}_{h_i}$, and $\vec{u}_{i-1} + \cdots + \vec{u}_i^*$ is orthogonal to $\mathcal{H}_{h_i}$. Thus

   $$M_i(\vec{\tilde{u}}, \vec{v}) = a(\vec{u}_i, P_i H^i_{\Omega_{i+1}}\vec{v}) = a(\vec{u}_i, H^i_{\Omega_{i+1}}\vec{v}).$$

Since $\vec{v} \in \mathcal{H}_{h_0} + \cdots + \mathcal{H}_{h_i}$, it follows that

$$\vec{v} - H^i_{\Omega_{i+1}} \vec{v} \in \mathcal{H}_{h_i} \cap \mathcal{H}_{h_{i+1}},$$

which is orthogonal to $\vec{u}_i$. Thus, we obtain

$$M_i(\vec{\bar{u}}, \vec{v}) = a(\vec{u}_i, (P_i H^i_{\Omega_{i+1}} \vec{v} - \vec{v}) + \vec{v}) = a(\vec{u}_i, \vec{v}).\Box$$

We now present the definition of the cosine of the angle between spaces, and apply it to the case of two level FAC methods to get a bound for the spectral radius of the error propagation map. See Mandel and McCormick [24] and Maitre and Musy [23].

DEFINITION. Let U and V denote two subspaces of a Hilbert space $\mathcal{H}$ with inner product $a(.,.)$. We define the cosine of the angle between U and V as:

$$cos(U,V) \equiv \sup_{u \in U, v \in V, u, v \neq 0} \frac{|a(u,v)|}{\sqrt{a(u,u)a(v,v)}}.$$

Note that $cos(U,V) \in [0,1]$, and if $U \cap V$ is nontrivial, then $cos(U,V) = 1$. If U and V are orthogonal, then $cos(U,V) = 0$. Furthermore, if $u \in U, v \in V$, then

$$a(u,v) \leq cos(U,V)\sqrt{a(u,u)}\sqrt{a(v,v)}.$$

When $cos(U,V) < 1$, this is often referred to as a *strengthened Cauchy inequality*. For a proof of the following two Lemmas see Mandel and McCormick [24].

Lemma 24 *Let U and V be nontrivial subspaces of a Hilbert space $\mathcal{H}$. Then*

$$\rho(P_U P_V) = ||P_U P_V P_U|| = cos^2(U,V).$$

where P_U, P_V denote the orthogonal projections onto the respective spaces, and $\rho(P_U P_V)$ denotes the spectral radius. The norm is the Hilbert space operator norm.

Lemma 25 *Let U and V be nontrivial subspaces of $\mathcal{H}$.*

1. *If $U \cap V = \{0\}$, then $cos(U^\perp, V^\perp) \geq cos(U,V)$.*
2. *If $\mathcal{H} = U \oplus V$, then $cos(U^\perp, V^\perp) = cos(U,V)$.*
3. *If $U^* \subset U$ and $V^* \subset V$ and $\mathcal{H} = U^* \oplus V^*$, then $cos(U^\perp, V^\perp) \leq cos(U^*, V^*)$.*

We can use the preceeding two Lemmas to estimate the spectral radius of the two level FAC method, on $\mathcal{H}_{h_0,h_1}$ using the subspaces $\mathcal{H}_{h_0}$ and $\mathcal{H}_{h_1}$. We let

$$U^* = U = \mathcal{H}_{h_0},$$

and

$$V^* = \{w = (I - \Pi^{h_0})v : v \in \mathcal{H}_{h_1}\} \subset V = \mathcal{H}_{h_1},$$

where Π^{h_0} is the coarse mesh Raviart-Thomas interpolation map, as defined in Chapter 1. Thus, the spectral radius

$$\rho(P^{\perp}_{\mathcal{H}_{h_1}} P^{\perp}_{\mathcal{H}_{h_0}}) = cos^2(\mathcal{H}^{\perp}_{h_0}, \mathcal{H}^{\perp}_{h_1}) \leq cos^2(\mathcal{H}_{h_0}, (I - \Pi^{h_0})\mathcal{H}_{h_1}).$$

To obtain a bound for $cos(\mathcal{H}_{h_0}, (I - \Pi^{h_0})\mathcal{H}_{h_1})$, we work on each coarse mesh element separately, and sum over all the coarse mesh elements. Thus suppose that for each element $K \in \tau^{h_0}(\Omega_1)$, there exists a non-negative constant

$$0 \leq \sigma_K \leq 1,$$

such that

$$a_K(u,v) \leq \sigma_K \sqrt{a_K(u,u)}\sqrt{a_K(v,v)}, \quad \forall u \in \mathcal{H}_{h_0}, \text{ and } \forall v \in (I - \Pi^{h_0})\mathcal{H}_{h_1}.$$

Then

$$\begin{aligned} a(u,v) = \sum_K a_K(u,v) &\leq \sum_K \sigma_K \sqrt{a_K(u,u)}\sqrt{a_K(v,v)} \\ &\leq (\max_K \sigma_K)\sqrt{a(u,u)}\sqrt{a(v,v)}. \end{aligned}$$

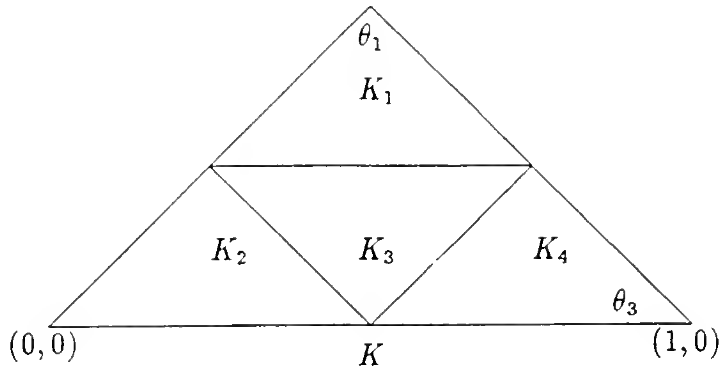
Thus,

$$cos(\mathcal{H}_{h_0}, (I - \Pi^{h_0})\mathcal{H}_{h_1}) \leq (\max_K \sigma_K),$$

where σ_K is $cos(\mathcal{H}_{h_0}|_K, (I - \Pi^{h_0})\mathcal{H}_{h_1}|_K)$. We present here the results of an explicit computation of such a constant, using techniques presented in Maitre and Musy [23]. We consider the particular case of the lowest order Raviart-Thomas spaces, i.e., $r = 0$, defined on a mesh of triangles.

Let K denote a triangle in the coarse mesh $\tau^{h_0}(\Omega_1)$, with vertices a_1, a_2, a_3. Let the angle at each vertex be donoted by $\theta_1, \theta_2, \theta_3$, respectively. We consider the refinement $\tau^{h_1}(\Omega_1)$ obtained by dividing each coarse triangle K into four equivalent subtriangles by connecting the midpoint of each edge of K. We denote the subtriangles by $K_1, \ldots, K_4$. See Figure 3.2.

Figure 3.2: Standard refinement of a triangle, used to compute the angle between coarse and fine mesh spaces.



Recall that a function in $X_{h_0}(\Omega_0)$ for $r = 0$, restricted to K, is specified by the three values of the normal component on the mid-point of edges. But for functions in $\mathcal{H}_{h_0}$, the *divergence free* constraint reduces the number to two. Thus, the dimension of $\mathcal{H}_{h_0}|_K$ is two. A basis for the two linearly independent *divergence free* functions in $\mathcal{H}_{h_0}|_K$ is given by:

$$\vec{u}_1 = (1,0) \text{ and } \vec{u}_2 = (0,1).$$

There are nine edges amongst the four subtriangles, and there are four constraints giving us five linearly independent *divergence free* functions defined on $\mathcal{H}_{h_1}$ restricted to K. Since two of them belong to $\mathcal{H}_{h_0}$, there are only three basis for functions in $(I - \Pi^{h_0})\mathcal{H}_{h_1}|_K$. They are given by:

$$\vec{u}_3 = \begin{cases} \left(\frac{1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}\right) & \text{on } K_1 \\ \left(\frac{-1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}\right) & \text{on } K_2 \\ \left(\frac{-1-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)}{2}\right) & \text{on } K_3 \\ \left(\frac{-1+\cos(\theta_2)}{2}, \frac{\sin(\theta_2)}{2}\right) & \text{on } K_4 \end{cases}$$

$$\vec{u}_4 = \begin{cases} \left(\frac{\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)-\sin(\theta_3)}{2}\right) & \text{on } K_1 \\ \left(\frac{-\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)+\sin(\theta_3)}{2}\right) & \text{on } K_2 \\ \left(\frac{\cos(\theta_3)-\cos(\theta_2)}{2}, \frac{-\sin(\theta_2)-\sin(\theta_3)}{2}\right) & \text{on } K_3 \\ \left(\frac{\cos(\theta_3)+\cos(\theta_2)}{2}, \frac{\sin(\theta_2)-\sin(\theta_3)}{2}\right) & \text{on } K_4 \end{cases}$$

and

$$\vec{u}_5 = \begin{cases} \left(\frac{\cos(\theta_3)+1}{2}, \frac{-\sin(\theta_3)}{2}\right) & \text{on } K_1 \\ \left(\frac{-\cos(\theta_3)-1}{2}, \frac{\sin(\theta_3)}{2}\right) & \text{on } K_2 \\ \left(\frac{\cos(\theta_3)-1}{2}, \frac{-\sin(\theta_3)}{2}\right) & \text{on } K_3 \\ \left(\frac{\cos(\theta_3)-1}{2}, \frac{-\sin(\theta_3)}{2}\right) & \text{on } K_4 \end{cases}$$

Computing the local stiffness matrix on K using the basis $\vec{u}_1, \ldots, \vec{u}_5$, we obtain a block partitioned matrix of the form:

$$\begin{bmatrix} M_K & W_K \\ W_K^T & N_K \end{bmatrix},$$

where

$$\begin{cases} (M_K)_{ij} &=& a_K(\vec{u}_i, \vec{u}_j); \quad i,j = 1,2 \\ (W_K)_{ij} &=& a_K(\vec{u}_i, \vec{u}_j); \quad i = 1,2; \; j = 3,4,5 \\ (N_K)_{ij} &=& a_K(\vec{u}_i, \vec{u}_j); \quad i,j = 3,4,5. \end{cases}$$

On K, using Lemma 24, we can estimate $cos^2(\mathcal{H}_{h_0}|_K, (I - \Pi^{h_0})\mathcal{H}_{h_1}|_K)$ by the spectral radius of the error propagation map:

$$cos^2(\mathcal{H}_{h_0}|_K, (I - \Pi^{h_0})\mathcal{H}_{h_1}|_K)$$

$$= \rho\left(\left(I - \begin{bmatrix} M_K^{-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} M_K & W_K \\ W_K^T & N_K \end{bmatrix}\right)\left(I - \begin{bmatrix} 0 & 0 \\ 0 & N_K^{-1} \end{bmatrix} \begin{bmatrix} M_K & W_K \\ W_K^T & N_K \end{bmatrix}\right)\right).$$

This is the same as

$$\rho(M_K^{-1} W_K N_K^{-1} W_K^T) = \rho(N_K^{-1} W_K^T M_K^{-1} W_K).$$

In the case of the Laplacian, where

$$\mathcal{A}(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } a_K(\vec{u}, \vec{v}) = \int_K \vec{u}^T \vec{v} dx,$$

we obtain,

$$M_K = \frac{S}{4}\begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}; \quad W_K = \frac{S}{4}\begin{bmatrix} -1 - \cos(\theta_2) & \cos(\theta_3) - \cos(\theta_2) & \cos(\theta_3) - 1 \\ -\sin(\theta_2) & -\sin(\theta_2) - \sin(\theta_3) & -\sin(\theta_3) \end{bmatrix},$$

and

$$N_K = \frac{S}{4}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Here S denotes the area of triangle K. Furthermore,

$$M_K^{-1} W_K N_K^{-1} W_K^T$$

$$= \frac{1}{4}\begin{bmatrix} 1+\cos^2(\theta_2)+\cos^2(\theta_3) & \sin(\theta_2)\cos(\theta_2)-\sin(\theta_3)\cos(\theta_3) \\ \sin(\theta_2)\cos(\theta_2)-\sin(\theta_3)\cos(\theta_3) & \sin^2(\theta_2)+\sin^2(\theta_3) \end{bmatrix},$$

and its spectral radius is

$$\rho = \frac{\frac{3}{4}+\sqrt{\frac{9}{16}-\frac{4}{16}d(\theta_1,\theta_2)}}{2},$$

where

$$d(\theta_1,\theta_2) \equiv 2\sin^2(\theta_2)\cos^2(\theta_3)+2\sin^2(\theta_3)+\frac{1}{2}\sin(2\theta_2)\sin(2\theta_3).$$

For an equilateral triangle

$$\rho = \frac{3}{8},$$

and for a right triangle

$$\rho = \frac{1}{2}.$$

3.2.3 Quantitative bounds for some many level FAC algorithms.

Now that we have computed the spectral radius of a two level FAC method, we consider an application to the case of the many level FAC method using the same lowest order Raviart-Thomas refined mesh spaces based on triangles. First, we present a proof of Lemma 19, for the FAC algorithm. Then, using this Lemma and the bound for the two subspace problems, we give a bound independent of the geometry.

Recall that the preconditioner corresponding to the *symmetrised* FAC algorithm is given by

$$\tilde{a}(\vec{u},\vec{v}) = a(P_l\vec{u},P_l\vec{v}) + \sum_{i=1}^{n} a(P_i H^i_{\Omega_{i+1}}\vec{u}, P_i H^i_{\Omega_{i+1}}\vec{v}).$$

Note that

$$a(\vec{u},\vec{v}) = a(P_l\vec{u},P_l\vec{u}) + a(P_l^\perp\vec{u},P_l^\perp\vec{u}).$$

We can form a preconditioner by replacing $P_l^\perp$ by $H^{l-1}_{\Omega_l}$. We obtain:

$$\tilde{a}_2(\vec{u},\vec{v}) \equiv a(P_l\vec{u},P_l\vec{v}) + a(H^{l-1}_{\Omega_l}\vec{u},H^{l-1}_{\Omega_l}\vec{v}).$$

Because $H^{l-1}_{\Omega_l}\vec{u}$ is the $\mathcal{H}_{l-1}$ harmonic extension of $\vec{u}$ into Ω_l, and because $P_l^\perp\vec{u}-H^{l-1}_{\Omega_l}\vec{u}$ is $a(.,.)$-orthogonal to $P_l^\perp\vec{u}$, we obtain that

$$a(H^{l-1}_{\Omega_l}\vec{u},H^{l-1}_{\Omega_l}\vec{u}) \geq a(P_l^\perp\vec{u},P_l^\perp\vec{u}).$$

Thus,

$$\tilde{a}_2(\vec{u},\vec{u}) \geq a(\vec{u},\vec{u}).$$

Note that, $\tilde{a}_2(.,.)$ is the preconditioner corresponding to a two subspace *symmetrised* multiplicative Schwarz algorithm on $\mathcal{H}_{h_0,\ldots,h_l}$ involving the subspaces $\mathcal{H}_{h_l}$ and $\sum_{i=0}^{l-1}\mathcal{H}_{h_i}$. Thus, we would require solvers to determine the projections onto $\mathcal{H}_{h_l}$ and onto $\sum_{i=0}^{l-1}\mathcal{H}_{h_i}$.

We could apply the same technique to the $a(H_{\Omega_l}^{l-1}\vec{u}, H_{\Omega_l}^{l-1}\vec{u})$ term. i.e., decompose

$$H_{\Omega_l}^{l-1} = P_{l-1}H_{\Omega_l}^{l-1} + P_{l-1}^{\perp}H_{\Omega_l}^{l-1},$$

and then replace $P_{l-1}^{\perp}H_{\Omega_l}^{l-1}$ by $H_{\Omega_{l-1}}^{l-2}H_{\Omega_l}^{l-1}$. The resulting preconditioner, which we denote by $\tilde{a}_3(.,.)$ is:

$$\tilde{a}_3(\vec{u},\vec{v}) \equiv a(P_l\vec{u}, P_l\vec{v}) + a(P_{l-1}H_{\Omega_l}^{l-1}\vec{u}, P_{l-1}H_{\Omega_l}^{l-1}\vec{v}) + a(H_{\Omega_{l-1}}^{l-2}H_{\Omega_l}^{l-1}\vec{u}, H_{\Omega_{l-1}}^{l-2}H_{\Omega_l}^{l-1}\vec{v}).$$

Note that $H_{\Omega_{l-1}}^{l-2}H_{\Omega_l}^{l-1}$ is the same as $H_{\Omega_{l-1}}^{l-2}$. Thus,

$$\tilde{a}_3(\vec{u},\vec{u}) \geq \tilde{a}_2(\vec{u},\vec{u}) \geq a(\vec{u},\vec{u}).$$

Similarly, for $3 \leq j \leq l+1$, we define

$$\tilde{a}_j(\vec{u},\vec{v}) \equiv a(P_l\vec{u}, P_l\vec{v}) + \sum_{i=1}^{j-2} a(P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{u}, P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{v}) + a(H_{\Omega_{l-j+2}}^{l-j+1}\vec{u}, H_{\Omega_{l-j+2}}^{l-j+1}\vec{v}).$$

Note that

$$\tilde{a}_{l+1}(\vec{u},\vec{v}) = \tilde{a}(\vec{u},\vec{v}).$$

To prove Lemma 19, we consider:

$$\frac{\tilde{a}_{l+1}(\vec{u},\vec{v})}{a(\vec{u},\vec{u})}$$

$$= \frac{\tilde{a}_{l+1}(\vec{u},\vec{v})}{\tilde{a}_l(\vec{u},\vec{u})}\frac{\tilde{a}_l(\vec{u},\vec{v})}{\tilde{a}_{l-1}(\vec{u},\vec{u})}\cdots\frac{\tilde{a}_3(\vec{u},\vec{v})}{\tilde{a}_2(\vec{u},\vec{u})}\frac{\tilde{a}_2(\vec{u},\vec{v})}{a(\vec{u},\vec{u})}.$$

Note that, by construction,

$$1 \leq \frac{\tilde{a}_{l+1}(\vec{u},\vec{v})}{a(\vec{u},\vec{u})}.$$

To find an upper bound, we consider each term in the product, and bound it by using techniques for the two level FAC methods:

$$\frac{\tilde{a}_{j+1}(\vec{u},\vec{u})}{\tilde{a}_j(\vec{u},\vec{u})}$$

$$= \frac{a(P_l\vec{u}, P_l\vec{v}) + \sum_{i=1}^{j-1} a(P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{u}, P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{v}) + a(H_{\Omega_{l-j+1}}^{l-j}\vec{u}, H_{\Omega_{l-j+1}}^{l-j}\vec{v})}{a(P_l\vec{u}, P_l\vec{v}) + \sum_{i=1}^{j-2} a(P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{u}, P_{l-i}H_{\Omega_{l-i+1}}^{l-i}\vec{v}) + a(H_{\Omega_{l-j+2}}^{l-j+1}\vec{u}, H_{\Omega_{l-j+2}}^{l-j+1}\vec{v})}.$$

Since only the last terms of $\tilde{a}_{j+1}(.,.)$ and $\tilde{a}_j(.,.)$ differ,

$$\frac{\tilde{a}_{j+1}(\vec{u}, \vec{u})}{\tilde{a}_j(\vec{u}, \vec{u})}$$

$$\leq \frac{a(P_{l-j+1}H_{\Omega_{l-j+2}}^{l-j+1}\vec{u}, P_{l-j+1}H_{\Omega_{l-j+2}}^{l-j+1}\vec{u}) + a(H_{\Omega_{l-j+1}}^{l-j}\vec{u}, H_{\Omega_{l-j+1}}^{l-j}\vec{u})}{a(H_{\Omega_{l-j+2}}^{l-j+1}\vec{u}, H_{\Omega_{l-j+2}}^{l-j+1}\vec{u})}.$$

Letting $\vec{w} = H_{\Omega_{l-j+2}}^{l-j+1}\vec{u} \in \mathcal{H}_0 + \cdots + \mathcal{H}_{l-j+1}$, we need a bound for

$$\frac{a(P_{l-j+1}\vec{w}, P_{l-j+1}\vec{w}) + a(H_{\Omega_{l-j+1}}^{l-j}\vec{w}, H_{\Omega_{l-j+1}}^{l-j}\vec{w})}{a(\vec{w}, \vec{w})}.$$

This can be bounded by the condition number of the two level *symmetrised* FAC algorithm applied to $\mathcal{H}_0 + \cdots + \mathcal{H}_{l-j+1}$, with the subspaces $\mathcal{H}_{l-j+1}$ and $\mathcal{H}_0 + \cdots + \mathcal{H}_{l-j}$. This completes the proof of Lemma 19, in the case of iterative refinement methods.

We now consider a quantitative bound for the many level *symmetrised* FAC algorithm for the case of the lowest order Raviart-Thomas finite element spaces based on triangles. We assume that the refinements in the appropriate regions are obtained by dividing each coarse mesh triangle into four subtriangles in the standard way. Then, each of the two subspace *symmetrised* FAC has a condition number bounded by

$$\kappa_i \leq \frac{1}{1-\rho}.$$

Thus, the condition number of the $l+1$ level *symmetrised* FAC is bounded, as in Lemma 19, by

$$\left[\frac{1}{1-\rho}\right]^l.$$

For, a mesh consisting of right triangles, the spectral radius of the *symmetrised* FAC method is $\rho = \frac{1}{4}$, and thus $\kappa_i = \frac{4}{3}$, thus our bound for the $l+1$ level *symmetrised* FAC method in this case is given by, $\left(\frac{4}{3}\right)^l$.

For the case of a mesh containing only equilateral triangles, $\rho = \frac{9}{64}$, and $\kappa = \frac{64}{55}$, and the condition number of the *symmetrised* $l+1$ level FAC method is bounded by $\left(\frac{64}{55}\right)^l$.

3.2.4 Step 3: Determining the pressure.

Let $(\vec{u}_h, p_h)$ denote the exact solution of discrete problem (3.2). Then, by construction,

$$\vec{u}_h = \vec{u}_{h,I} + \vec{u}_{h,DF},$$

which was computed in the previous two steps. Recall that, for the mixed formulation of elliptic Neumann problems, $W_h = 0$, and so p_h satisfies the following equation:

$$\left\{\begin{array}{llll} \text{Find } p_h \in Y_{h_0,\ldots,h_l}(\Omega) \text{ such that} & & & \\ a(\vec{u}_{h,I} + \vec{u}_{h,DF}, \vec{v}) + b(\vec{v}, p_h) & = & 0 & \forall \vec{v} \in X_{h_0,\ldots,h_l}(\Omega) \\ b(\vec{u}_{h,I} + \vec{u}_{h,DF}, q) & = & \int_\Omega f(x)q(x)dx & \forall q \in Y_{h_0,\ldots,h_l}. \end{array}\right. \tag{3.14}$$

Now, we can find the projection onto the *divergence free* Raviart-Thomas subspaces $\mathcal{H}_{h_i}$ by substituting the new residual into equation (3.14). For $i = 0, \ldots, l$ we solve:

$$\left\{\begin{array}{llll} \text{Find } \vec{w}_{h_i} \in X_{h_i}(\Omega_i), p_{h_i} \in Y_{h_i}(\Omega_i), \text{ such that} & & & \\ a(\vec{w}_{h_i}, \vec{v}) + b(\vec{v}, p_{h_i}) & = & -a(\vec{u}_{h,I} + \vec{u}_{h,DF}, \vec{v}) = b(\vec{v}, p_h), & \forall \vec{v} \in X_{h_i}(\Omega_i) \\ b(\vec{w}_{h_i}, q) & = & 0, & \forall q \in Y_{h_i}(\Omega_i). \end{array}\right. \tag{3.15}$$

Each $\vec{w}_{h_i} = 0$, but p_{h_i} will be nonzero in general. Using the fact that the *divergence* operator maps $X_{h_i}(\Omega_i)$ onto $Y_{h_i}(\Omega_i)$, we see that

$$\int_{\Omega_i} (p_h - p_{h_i}) q dx = 0, \quad \forall q \in Y_{h_i}(\Omega_i),$$

i.e., p_{h_i} is the L^2 projection of the discrete pressure solution p_h onto $Y_{h_i}(\Omega_i)$. Recall that

$$Y_{h_i}(\Omega_i) \subset L^2(\Omega_i)/\mathcal{P}_0(\Omega_i),$$

implying that p_{h_i} has mean value zero on Ω_i. Thus, if we knew the mean value of p_h on Ω_i, then we would know the L^2 projection of p_h onto $Y_{h_i}(\Omega_i) \oplus R$, where R denotes the real numbers. For each i, let $\overline{p_{\Omega_i}}$ denote the mean value of p_h on Ω_i. i.e.,

$$\overline{p_{\Omega_i}} \equiv \frac{\int_{\Omega_i} p_h dx}{\int_{\Omega_i} dx}.$$

We then obtain

$$\int_{\Omega_i} (p_{h_i} + \overline{p_{\Omega_i}} - p_h) q dx = 0, \quad \forall q \in Y_{h_i}(\Omega_i) \oplus \mathcal{P}_0(\Omega_i),$$

i.e., $p_{h_i} + \overline{p_{\Omega_i}}$ is the L^2 projection of p_h onto $Y_{h_i}(\Omega_i) \oplus R$. Using this, we can compute the mean value of p_h on Ω_{i+1}:

$$\int_{\Omega_{i+1}} p_h dx = \int_{\Omega_{i+1}} (p_{h_i} + \overline{p_{\Omega_i}}) dx \Longrightarrow \overline{p_{\Omega_{i+1}}} = \frac{\int_{\Omega_{i+1}} (p_{h_i} + \overline{p_{\Omega_i}}) dx}{\int_{\Omega_{i+1}} dx}.$$

Using the fact that the discrete pressures are discontinuous across element boundaries, and the property that functions in

$$Y_{h_0,\dots,h_l}(\Omega) \oplus \mathcal{P}_0(\Omega),$$

when restricted to $\Omega_i - \Omega_{i+1}$, lie in

$$[Y_{h_i}(\Omega_i) \oplus \mathcal{P}_0(\Omega_i)]|_{\Omega_i - \Omega_{i+1}},$$

we obtain that

$$p_{h_i}(x) + \overline{p_{\Omega_i}} = p_h(x), \quad \forall x \in \Omega_i - \Omega_{i+1}.$$

The same is true for $i = l$, when $\Omega_{l+1} \equiv \emptyset$. Thus, if p_{h_i} and $\overline{p_{\Omega_i}}$ are known for $i = 0, \dots, l$, we can construct p_h:

$$\begin{cases} \text{For } i = 0, \dots, l \\ p_h = p_{h_i} + \overline{p_{\Omega_i}}, \qquad \text{on } \Omega_i - \Omega_{i+1}, \\ \text{and } \overline{p_{\Omega_{i+1}}} = \dfrac{\int_{\Omega_{i+1}} (p_{h_i} + \overline{p_{\Omega_i}})dx}{\int_{\Omega_{i+1}} dx}. \end{cases}$$

But, for $i = 0$, the discrete pressure p_{h_0}, which is defined on the whole domain Ω, has the same mean value as p_h on Ω, namely, zero. Thus, using the above procedure, and the L^2 projections, p_{h_i}, we can construct p_h.

Chapter 4

A Dirichlet-Neumann algorithm for Raviart-Thomas elements.

Let Ω be a polygonal domain in R^2 which is triangulated by a *uniformly shape regular* triangulation $\tau^h(\Omega)$. Let

$$\begin{cases} \Omega &= \Omega_1 \cup \Omega_2 \cup \Gamma, \text{ where } \Omega_1 \cap \Omega_2 = \emptyset, \\ \Gamma &\equiv \partial\Omega_1 \cap \partial\Omega_2, \end{cases}$$

where Ω_1 and Ω_2 are connected, mutually disjoint subdomains. We assume that each Ω_i is a union of elements in the triangulation $\tau^h(\Omega)$. See Figure 4.1.

As in earlier Chapters, we use $X_h(\Omega) \subset \vec{H}_{0,\partial\Omega}(div,\Omega)$ and $Y_h(\Omega) \subset L^2(\Omega)/\mathcal{P}_0(\Omega)$ to denote the Raviart-Thomas spaces defined on the triangulation $\tau^h(\Omega)$, for an elliptic problem with Neumann boundary conditions. Recall that $\mathcal{P}_0(\Omega)$ denotes the space of constants on Ω. We decompose

$$X_h(\Omega) = X_1 \oplus X_2 \oplus X_3 \oplus X_4,$$

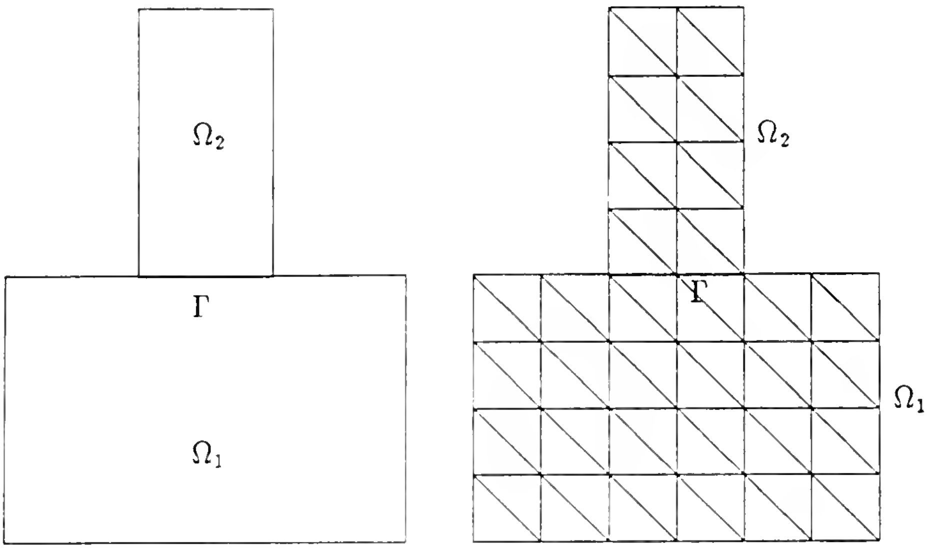
and

$$Y_h(\Omega) \oplus \mathcal{P}_0(\Omega) = Y_1 \oplus Y_2 \oplus Y_3 \oplus Y_4,$$

where the subspaces are defined by:

$$\begin{array}{lclcl} X_1 &\equiv& X_h(\Omega_1) &=& X_h(\Omega) \cap \vec{H}_{0,\partial\Omega_1}(div,\Omega_1) \\ X_2 &\equiv& X_h(\Omega_2) &=& X_h(\Omega) \cap \vec{H}_{0,\partial\Omega_2}(div,\Omega_2) \\ Y_1 &\equiv& Y_h(\Omega_1) &=& Y_h(\Omega) \cap L^2(\Omega_1)/\mathcal{P}_0(\Omega_1) \\ Y_2 &\equiv& Y_h(\Omega_2) &=& Y_h(\Omega) \cap L^2(\Omega_2)/\mathcal{P}_0(\Omega_2) \\ Y_3 &\equiv& \mathcal{P}_0(\Omega_1) && \\ Y_4 &\equiv& \mathcal{P}_0(\Omega_2). && \end{array} \tag{4.1}$$

Figure 4.1: An example of a 'T' shaped region.



Note that the *dimension* of Y_3 as well as of Y_4 is one. Before we define X_3 and X_4 we define $X_h(\Gamma)$, a complementary subspace of $X_1 \oplus X_2$ in $X_h(\Omega)$:

$$X_h(\Gamma) \equiv \{\vec{u}_h \in X_h(\Omega) : \Pi^h(\Omega_i)\vec{u}_h = 0, \text{ for } i = 1,2\},$$

where $\Pi^h(\Omega_i)$ denotes the interpolation map onto $X_h(\Omega_i)$. Let $\vec{\pi_0}$ be a function in $X_h(\Gamma)$ which has a *normal trace* of non-zero mean value on Γ. i.e.,

$$\int_\Gamma \gamma_n \vec{\pi_0} ds_x \neq 0. \tag{4.2}$$

For instance, we could define $\vec{\pi_0}$ to have a *normal trace* of 1 on Γ. We then define

$$X_3 \equiv \{c\vec{\pi_0} : c \in R\}.$$

Note that the *dimension* of X_3 is 1. Next, we define

$$X_4 \equiv \{\vec{u}_h \in X_h(\Gamma) : \int_\Gamma \gamma_n \vec{u}_h ds_x = 0\}.$$

Thus,

$$X_h(\Gamma) = X_3 \oplus X_4.$$

Having described the subspaces $X_1, \ldots, Y_4$, we can use them to obtain a discretisation of problem (1.7). By choosing a basis for each of the subspaces and ordering

them, we obtain the following block partitioned stiffness matrix:

$$
\begin{bmatrix}
A_{11} & 0 & A_{13} & A_{14} & B_{11}^T & 0 & 0 & 0 \\
0 & A_{22} & A_{23} & A_{24} & 0 & B_{22}^T & 0 & 0 \\
A_{13}^T & A_{23}^T & A_{33} & A_{34} & B_{13}^T & B_{23}^T & B_{33}^T & B_{43}^T \\
A_{14}^T & A_{24}^T & A_{34}^T & A_{44} & B_{14}^T & B_{24}^T & 0 & 0 \\
B_{11} & 0 & B_{13} & B_{14} & 0 & 0 & 0 & 0 \\
0 & B_{22} & B_{23} & B_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & B_{33} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B_{43} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix}
=
\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \end{bmatrix}
\tag{4.3}
$$

The unknowns in X_i are represented by u_i, and those in Y_i are represented by p_i, in the matrix notation. We note that $A_{12} = 0$, since functions in X_1 and X_2 have support in Ω_1 and Ω_2 respectively, which are mutually disjoint. Similarly, B_{12}, B_{21} are zero submatrices. Next, we note that since Y_3 and Y_4 consist of piecewise constant pressures, and since the velocities in X_1, X_2, and X_4 have *normal traces* which have mean value zero on each $\partial\Omega_i$, we obtain that $B_{31}, B_{41}, B_{32}, B_{42}, B_{34}, B_{44}$ are zero submatrices.

REMARK. Since we have included $\mathcal{P}_0(\Omega)$ (the constants) in the pressure spaces, the stiffness matrix will have a null space of *dimension* one corresponding to a constant pressure.

REMARK. The linear system (4.3), has the form

$$
\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}
\begin{bmatrix} u \\ p \end{bmatrix}
=
\begin{bmatrix} 0 \\ F \end{bmatrix}.
$$

4.1 The Glowinski-Wheeler algorithm.

In this section, we discuss a domain decomposition algorithm due to Glowinski and Wheeler [16], to solve equation (4.3), and later, in the next section, we describe a *Dirichlet-Neumann* preconditioner. We describe their algorithm, [16], only for the case of *two* non-overlapping subdomains, though their algorithm is applicable to the case of arbitrarily many non-overlapping subdomains. So far, the Dirichlet-Neumann preconditioner introduced here, holds only for the case of *two* subdomains. Each iteration of the Glowinsk-Wheeler algorithm, involves the solution of a saddle point problem on each of the two subdomains.

The Glowinski-Wheeler algorithm is divided into two steps. Let $(\vec{u}_h, p_h)$ denote

the solution of problem (4.3). We decompose the discrete solution as

$$(\vec{u}_h, p_h) = (\vec{u}_{h,I}, p_{h,I}) + (\vec{u}_{h,H}, p_{h,H}),$$

where I denotes *initial* or *inhomogeneous* and H denotes *pieciewise discrete harmonic* and *divergence free*. In the first step of the Glowinski-Wheeler algorithm we determine $(\vec{u}_{h,I}, p_{h,I})$. This is equivalent to the saddle point problem being reduced to a symmetric positive definite Schur complement problem involving the unknowns in X_4, i.e., the velocity unknowns on the interface Γ, with mean value zero on Γ. In the second step we determine $(\vec{u}_{h,H}, p_{h,H})$, using an iterative method, such as the conjugate gradient algorithm, to solve the positive definite Schur complement problem. Throughout this Chapter, we use function and matrix notation interchangeably. For instance, we use the vector notation $[u_1, \ldots, p_4]$ to denote $(\vec{u}_h, p_h)$, etc.

To describe the algorithm, we reorder the unknowns as follows:

$$\begin{array}{l} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \\ (r6): \\ (r7): \\ (r8): \end{array} \begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ p_1 \\ u_2 \\ p_2 \\ u_3 \\ u_4 \\ p_3 \\ p_4 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_2 \\ 0 \\ 0 \\ F_3 \\ F_4 \end{bmatrix} \tag{4.4}$$

STEP 1. The first step in the Glowinski-Wheeler algorithm is to find an initial discrete velocity and pressure $(\vec{u}_{h,I}, p_{h,I}) \in X_h(\Omega) \times Y_h(\Omega)$, such that when this is substituted in equation (4.4), and the residual is computed, the residual is zero in all rows except possibly in row $r6$. To find $(\vec{u}_{h,I}, p_{h,I})$, we solve a series of problems on substructures.

To start, we compute Neumann boundary data on Γ and then solve local problems on Ω_1 and Ω_2, using the given $f(x)$ in the *divergence* constraint and the computed Neumann boundary data extended by zero outside Γ as the specified Neumann boundary conditions on $\partial\Omega_1$ and $\partial\Omega_2$ respectively. The Neumann boundary data must be compatible with the mean value of $f(x)$ on each Ω_i. After the two local problems are solved, we compute mean values for the pressures in Ω_1 and Ω_2 by solving a system of dimension one, which thereby reduces the residual to be zero with respect to $r5$. We outline the steps in more detail:

`Step 1a.` Define

$$c^* \equiv \frac{\int_{\Omega_1} f(x)dx}{\int_{\partial\Omega_1} \gamma_n \vec{\pi_0} ds_x}.$$

Then, $c^*\vec{\pi_0}$ will provide Neumann boundary data on Γ which is compatible with $f(x)$ on both Ω_i, since by assumption, $f(x)$ has mean value zero on Ω. i.e.,

$$\int_{\Omega_i} \left(f(x) - \nabla \cdot c^*\vec{\pi_0}\right) dx = 0, \quad \text{for each } \Omega_i.$$

`Step 1b.` Using $c^*\vec{\pi_0}$, we solve the following subproblems for $i = 1, 2$:

$$\left\{ \begin{array}{lcll} \multicolumn{4}{l}{\text{Find } \vec{u}_{i,I} \in X_h(\Omega_i),\ p_{i,I} \in Y_h(\Omega_i) \text{ such that}} \\ a(\vec{u}_{i,I}, \vec{v}) + b(\vec{v}, p_{i,I}) & = & -a(c^*\vec{\pi_0}, \vec{v}), & \forall \vec{v} \in X_h(\Omega_i) \\ b(\vec{u}_{i,I}, q) & = & F(q) - b(c^*\vec{\pi_0}, q), & \forall q \in Y_h(\Omega_i) \end{array} \right. \tag{4.5}$$

By the construction of $c^*\vec{\pi_0}$, the local problems are well posed.

`Step 1c.` We choose the piecewise constants $p_{3,I} \in \mathcal{P}_0(\Omega_1)$, and $p_{4,I} \in \mathcal{P}_0(\Omega_2)$ such that

$$p_{3,I} \int_{\Omega_1} dx + p_{4,I} \int_{\Omega_2} dx = 0, \tag{4.6}$$

and

$$a(c^*\vec{\pi_0} + \vec{u}_{1,I} + \vec{u}_{2,I}, \vec{v}) + b(\vec{v}, p_{1,I} + p_{2,I} + p_{3,I} + p_{4,I}) = 0, \quad \text{for } \vec{v} = \vec{\pi_0}. \tag{4.7}$$

We define

$$\vec{u}_{h,I} \equiv c^*\vec{\pi_0} + \vec{u}_{1,I} + \vec{u}_{2,I},$$

and

$$p_{h,I} \equiv p_{1,I} + p_{2,I} + p_{3,I} + p_{4,I}.$$

Note that, by using equation (4.5), and the fact that the discrete pressures are discontinuous across element edges, we obtain that

$$B_h \vec{u}_{h,I} = F_h.$$

We now give a block Gaussian elimination interpretation of step (1). We use

$$[u_{1,I}, p_{1,I}, u_{2,I}, p_{2,I}, u_{3,I}, u_{4,I}, p_{3,I}, p_{4,I}]^T$$

to denote the matrix representation of $(\vec{u}_{h,I}, p_{h,I})$, obtained after the initial step. The equation satisfied by $(\vec{u}_{h,I}, p_{h,I})$, is:

$$\begin{array}{l} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \\ (r6): \\ (r7): \\ (r8): \end{array} \begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,I} \\ p_{1,I} \\ u_{2,I} \\ p_{2,I} \\ u_{3,I} \\ u_{4,I} \\ p_{3,I} \\ p_{4,I} \end{bmatrix} = \begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_2 \\ 0 \\ W_4 \\ F_3 \\ F_4 \end{bmatrix} \tag{4.8}$$

where $u_{4,I} \equiv 0$.

Finding c^* in step (1a) is equivalent to using either row $r7$ or $r8$ in equation (4.8) to solve for $u_{3,I}$. Equations $r7$ and $r8$ are equivalent since the matrix has a null space of *dimension* one spanned by values of p_3 and p_4 corresponding to a constant pressure on Ω. Note that the submatrices B_{33} and B_{43} are scalars, and nonzero. Thus,

$$u_{3,I} = \frac{F_3}{B_{33}} = \frac{F_4}{B_{43}}.$$

Solving the two local subproblems in step (1b) is equivalent to solving rows $r1, r2$ and $r3, r4$ in equation (4.8) for $u_{1,I}, p_{1,I}$ and $u_{2,I}, p_{2,I}$ respectively, by substituting in the values of $u_{3,I}$ and $u_{4,I}$ with $u_{4,I}$ *defined to be zero.* i.e.,

$$\begin{bmatrix} u_{i,I} \\ p_{i,I} \end{bmatrix} = \begin{bmatrix} A_{ii} & B_{ii}^T \\ B_{ii} & 0 \end{bmatrix}^{-1} \begin{bmatrix} -A_{i3}u_{3,I} \\ F_i - B_{i3}u_{3,I} \end{bmatrix},$$

for $i = 1, 2$. Thus, after steps (1a) and (1b), the equation satisfied by $(\vec{u}_{h,I}, p_{1,I} + p_{2,I})$, is:

$$\begin{array}{l} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \\ (r6): \\ (r7): \\ (r8): \end{array} \begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,I} \\ p_{1,I} \\ u_{2,I} \\ p_{2,I} \\ u_{3,I} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ F_1 \\ 0 \\ F_2 \\ W_3 \\ W_4 \\ F_3 \\ F_4 \end{bmatrix} \tag{4.9}$$

because $u_{4,I}$ is defined to be zero and $p_{3,I}$ and $p_{4,I}$ are so far undetermined.

`Step 1c.` Now, we note that scalars, $p_{3,I}$ and $p_{4,I}$ can be chosen so that row $r5$ in equation (4.8) is satisfied and so that the mean value of the pressure $p_{h,I}$ on Ω is

zero. This involves only the solution of two equations, because the block matrices, B_{33} and B_{43} are non-zero scalars, and the requirement that $p_{h,I}$ to have mean value zero is equivalent to equation (4.6), i.e.,

$$\left\{ \begin{array}{lcl} B_{33}^T p_{3,I} + B_{43}^T p_{4,I} & = & -\sum_{i=1}^{2}[A_{i3}^T u_{i,I} + B_{i3}^T p_{i,I}] - A_{33} u_{3,I}, \\ |\Omega_1| p_{3,I} + |\Omega_2| p_{4,I} & = & 0. \end{array} \right.$$

Since the last two columns of the matrix in equation (4.8) is nonzero only in the fifth row, the choice of $p_{3,I}$ and $p_{4,I}$ will not alter the right hand side, except in the fifth row, where, by the choice of $p_{3,I}$ and $p_{4,I}$ it is constructed to be zero. Thus W_4 would still be defined by the use of row $r6$ in equation (4.8), i.e.,

$$W_4 \equiv A_{14}^T u_{1,I} + B_{14}^T p_{1,I} + A_{24}^T u_{2,I} + B_{24}^T p_{2,I} + A_{34}^T u_{3,I}.$$

Thus, the residual after step (1), is given by

$$[0,0,0,0,0,-W_4,0,0]^T . \tag{4.10}$$

It is for such a right hand side, that we use iterative methods in the second step of the Glowinski-Wheeler algorithm. Define

$$(\vec{u}_{h,H}, p_{h,H}) \equiv (\vec{u}_h - \vec{u}_{h,I}, p_h - p_{h,I}).$$

Then, $(\vec{u}_{h,H}, p_{h,H})$ satisfies:

$$\begin{array}{c} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \\ (r6): \\ (r7): \\ (r8): \end{array} \begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,H} \\ p_{1,H} \\ u_{2,H} \\ p_{2,H} \\ u_{3,H} \\ u_{4,H} \\ p_{3,H} \\ p_{4,H} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -W_4 \\ 0 \\ 0 \end{bmatrix} \tag{4.11}$$

STEP 2. We now describe a Schur complement system which is equivalent to system (4.11), and involves the unknowns in $u_{4,H}$.

Step 2a. By using rows $r7$ or $r8$ in equation (4.11), we determine that

$$u_{3,H} = 0.$$

Step 2b. Following that, since $u_{3,H} = 0$, we can determine $u_{1,H}, p_{1,H}, u_{2,H}, p_{2,H}$ in terms of $u_{4,H}$, by using rows $r1, r2, r3, r4$ in equation (4.11). Namely,

$$\begin{bmatrix} u_{i,H} \\ p_{i,H} \end{bmatrix} = - \begin{bmatrix} A_{ii} & B_{ii}^T \\ B_{ii} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{i4} \\ B_{i4} \end{bmatrix} u_{4,H}, \quad \text{for } i = 1,2.$$

`Step 2c.` Since $u_{3,H} = 0$, once $u_{1,H}, p_{1,H}, u_{2,H}, p_{2,H}$ and $u_{4,H}$ are known, we can determine $p_{3,H}$ and $p_{4,H}$ of mean value zero on Ω by requiring that we obtain a right hand side of zero, in row $r5$ of equation (4.11), i.e., by solving:

$$\begin{cases} B_{33}^T p_{3,H} + B_{43}^T p_{4,H} &= -\sum_{i=1}^{2} [A_{i3}^T u_{i,H} + B_{i3}^T p_{i,H}] - A_{34} u_{4,H}, \\ |\Omega_1| p_{3,H} + |\Omega_2| p_{4,H} &= 0. \end{cases}$$

Thus far, we have not chosen $u_{4,H}$ so that row $r6$ of equation (4.11), is satisfied. Substituting the values of $u_{1,H}, p_{1,H}, u_{2,H}, p_{2,H}$ and $u_{3,H}$ in terms of $u_{4,H}$, into row $r6$ of equation (4.11), we see that $u_{4,H}$ must satisfy the following Schur complement system:

$$S_\Gamma u_{4,H} = -W_4, \tag{4.12}$$

where S_Γ is defined by

$$S_\Gamma \equiv A_{44} - \begin{bmatrix} A_{14} \\ B_{14} \end{bmatrix}^T \begin{bmatrix} A_{11} & B_{11}^T \\ B_{11} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{14} \\ B_{14} \end{bmatrix} - \begin{bmatrix} A_{24} \\ B_{24} \end{bmatrix}^T \begin{bmatrix} A_{22} & B_{22}^T \\ B_{22} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{24} \\ B_{24} \end{bmatrix}. \tag{4.13}$$

`Positive definiteness of` S_Γ. We now show that S_Γ is positive definite. Let $\tilde{u}_4$ be arbitrarily chosen, and let $\tilde{W}_4$ be defined by

$$\tilde{W}_4 \equiv S_\Gamma \tilde{u}_4.$$

Let $\tilde{u}_1, \tilde{p}_1, \ldots, \tilde{p}_4$ be constructed using $\tilde{u}_4$ such that:

$$\begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{p}_1 \\ \tilde{u}_2 \\ \tilde{p}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ \tilde{W}_4 \\ 0 \\ 0 \end{bmatrix} \tag{4.14}$$

for instance by using steps (2a), (2b), (2c). Then it follows that:

$$\tilde{u}_4^T S_\Gamma \tilde{u}_4 = \begin{bmatrix} \tilde{u}_1 \\ \tilde{p}_1 \\ \tilde{u}_2 \\ \tilde{p}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix}^T \begin{bmatrix} A_{11} & B_{11}^T & 0 & 0 & A_{13} & A_{14} & 0 & 0 \\ B_{11} & 0 & 0 & 0 & B_{13} & B_{14} & 0 & 0 \\ 0 & 0 & A_{22} & B_{22}^T & A_{23} & A_{24} & 0 & 0 \\ 0 & 0 & B_{22} & 0 & B_{23} & B_{24} & 0 & 0 \\ A_{13}^T & B_{13}^T & A_{23}^T & B_{23}^T & A_{33} & A_{34} & B_{33}^T & B_{43}^T \\ A_{14}^T & B_{14}^T & A_{24}^T & B_{24}^T & A_{34}^T & A_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & B_{43} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \tilde{u}_1 \\ \tilde{p}_1 \\ \tilde{u}_2 \\ \tilde{p}_2 \\ \tilde{u}_3 \\ \tilde{u}_4 \\ \tilde{p}_3 \\ \tilde{p}_4 \end{bmatrix}. \tag{4.15}$$

For nonzero $\tilde{u}_4$, the discrete velocity $[\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4]$ in equation (4.15) is non-zero and *divergence free.* Thus, using equation (4.15) and the fact that

$$\text{if } B^T u = 0 \text{ then } \begin{bmatrix} u \\ p \end{bmatrix}^T \begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix} \begin{bmatrix} u \\ p \end{bmatrix} = u^T A u,$$

we obtain that $\tilde{u}_4^T S_\Gamma \tilde{u}_4 > 0$, since A is positve definite on the *divergence free* space $\vec{H}(div^0, \Omega)$.

REMARK. We can compute the action of S_Γ on any vector $\tilde{u}_4$, by using matrix vector multiplications and by solving local subproblems. Thus, we can use the conjugate gradient algorithm to solve

$$S_\Gamma u_{4,H} = -W_4, \tag{4.16}$$

without actually computing the Schur complement S_Γ. Note that, once $u_{4,H}$ is known, the rest of the unknowns can be computed using steps (2a), (2b), and (2c), as we have already described. We refer to this basic algorithm as the Glowinski-Wheeler algorithm. Results of numerical tests using this algorithm, with a preconditioner, shows that the rate of convergence depends on the mesh width h, cf. Glowinski and Wheeler [16]. In the next section, we describe a *Dirichlet-Neumann* preconditioner, which we show, leads to a preconditioned system with a condition number independent of h.

4.2 A Dirichlet-Neumann preconditioner.

In this section we introduce a preconditioner to solve system (4.16) in the second step of the Glowinski-Wheeler algorithm, i.e., a preconditioner for the Schur complement S_Γ defined by equation (4.13). We follow ideas described in Bjørstad and Widlund [4] and Bramble, Pasciak and Schatz [5]; for related domain decomposition algorithms for the Stokes problem, see Pasciak [28] and Quarteroni [29]. The preconditioner will involve the solution of a saddle point problem on one of the subdomains Ω_i, with mixed Neumann and Dirichlet boundary conditions on the boundary $\partial\Omega_i$. Therefore, we first discuss the saddle point formulation of elliptic problems with mixed boundary conditions.

Consider the following elliptic problem for p on Ω_i:

$$\begin{cases} -\nabla \cdot (\mathcal{A}(x)\nabla p) &= 0 \quad \text{on } \Omega_i, \\ \vec{n} \cdot (\mathcal{A}(x)\nabla p) &= 0 \quad \text{on } \Gamma_i^c \equiv \partial\Omega_i - \Gamma, \\ p &= g \quad \text{on } \Gamma, \;\; g \in H_{00}^{1/2}(\Gamma). \end{cases}$$

The saddle point formulation of this problem is:

$$\begin{cases} \text{Find } \vec{u} \in \vec{H}_{0,\Gamma_i^c}(div,\Omega_i),\ p \in L^2(\Omega_i) \text{ such that} \\ a(\vec{u},\vec{v}) + b(\vec{v},p) \quad = \quad \int_\Gamma g\vec{n}\cdot\vec{v}ds_x, \quad \forall \vec{v} \in \vec{H}_{0,\Gamma_i^c},(div,\Omega_i) \\ b(\vec{u},q) \quad\quad\quad\quad = \quad 0, \quad\quad\quad\quad\quad \forall q \in L^2(\Omega_i), \end{cases} \tag{4.17}$$

where $a(\vec{u},\vec{v}) \equiv \int_\Omega \vec{v}^T\mathcal{A}(x)^{-1}\vec{u}dx$, and $b(\vec{v},p) \equiv \int_\Omega p\nabla\cdot\vec{v}dx$, as in the previous Chapters. Note that unlike the case of Neumann boundary conditions on the whole boundary $\partial\Omega_i$, the pressure p is unique in $L^2(\Omega_i)$. Also, since the Neumann data is prescribed only on Γ_i^c, the *normal trace* of $\vec{u} = \mathcal{A}(x)\nabla p$ is unknown on Γ. This will be reflected in the finite element discretisation of problem (4.17).

REMARK. Recall that for functions in $\vec{H}(div,\Omega_i)$ the *normal trace* is defined to be in $H^{-1/2}(\partial\Omega_i)$, by Lemma 4. Unfortunately, the restriction of a $H^{-1/2}(\partial\Omega_i)$ function to a subset of $\partial\Omega_i$, like $\bar{\Gamma} \subset \partial\Omega_i$, does not neccessarily lie in $H^{-1/2}(\bar{\Gamma})$. If that were not the case, then by using duality, we could show that $H^{1/2}(\bar{\Gamma})$ functions extended by zero outside, lie in $H^{1/2}(\partial\Omega_i)$. This, we know, is false.

REMARK. However, it can be shown that the restriction of a $H^{-1/2}(\partial\Omega_i)$ function restricted to $\bar{\Gamma}$ lies in $(H_{00}^{1/2}(\bar{\Gamma}))'$. This is because, we can define the action of the restriction on any function in $H_{00}^{1/2}(\bar{\Gamma})$ as follows:

$$\langle\gamma_n\vec{u}|_{\bar{\Gamma}},\phi\rangle \equiv \langle\gamma_n\vec{u},E^0\phi\rangle,$$

where $\phi \in H_{00}^{1/2}(\bar{\Gamma})$, and $E^0\phi \in H^{1/2}(\partial\Omega_i)$ is the *extension by zero* of ϕ. Recall that E^0 is a bounded map, by using the definition of $H_{00}^{1/2}(\bar{\Gamma})$. See Thomas [32]. Thus, the linear functional involving g in equation (4.17), is well defined.

REMARK. For any $\bar{\Gamma} \subset \partial\Omega_i$, the space $\vec{H}_{0,\bar{\Gamma}}(div,\Omega_i)$ is defined as:

$$\{\vec{u} \in \vec{H}(div,\Omega_i) : \langle\gamma_n\vec{u},\phi\rangle = 0,\ \forall\phi \in H_{00}^{1/2}(\bar{\Gamma})\}.$$

The discretisation of saddle point problem (4.17) is obtained by restricting the variational problem to appropriate Raviart-Thomas finite element subspaces of $\vec{H}_{0,\Gamma_i^c}(div,\Omega_i)$ and $L^2(\Omega_i)$, namely $V_h(\Omega_i) \cap \vec{H}_{0,\Gamma_i^c}(div,\Omega_i)$ and $Q_h(\Omega_i) \cap L^2(\Omega_i)$, respectively. Recall that $V_h(\Omega_i)$ and $Q_h(\Omega_i)$ denote the Raviart-Thomas spaces for the Dirichlet problem on Ω_i, as described in Chapter 1. For $i = 1$, we obtain the following linear system:

$$\begin{array}{l} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \end{array} \begin{bmatrix} A_{11} & B_{11}^T & A_{13} & A_{14} & 0 \\ B_{11} & 0 & B_{13} & B_{14} & 0 \\ A_{13}^T & B_{13}^T & A_{33}^{(1)} & A_{34}^{(1)} & B_{33}^T \\ A_{14}^T & B_{14}^T & A_{34}^{(1)\,T} & A_{44}^{(1)} & 0 \\ 0 & 0 & B_{33} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{1,H} \\ p_{1,H} \\ u_{3,H} \\ u_{4,H} \\ p_{3,H} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_4 \\ 0 \end{bmatrix} \tag{4.18}$$

For $i = 2$, we obtain:

$$\begin{array}{l} (r1): \\ (r2): \\ (r3): \\ (r4): \\ (r5): \end{array} \begin{bmatrix} A_{22} & B_{22}^T & A_{23} & A_{24} & 0 \\ B_{22} & 0 & B_{23} & B_{24} & 0 \\ A_{23}^T & B_{23}^T & A_{33}^{(2)} & A_{34}^{(2)} & B_{33}^T \\ A_{24}^T & B_{24}^T & A_{34}^{(2)\,T} & A_{44}^{(2)} & 0 \\ 0 & 0 & B_{33} & 0 & 0 \end{bmatrix} \begin{bmatrix} u_{2,H} \\ p_{2,H} \\ u_{3,H} \\ u_{4,H} \\ p_{4,H} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ G_4 \\ 0 \end{bmatrix} \qquad (4.19)$$

We have used the notation $A_{ij}^{(1)}$ to denote the appropriate submatrix in the stiffness matrix, where each entry is obtained by integrating the appropriate bilinear form only over subdomain Ω_1. Similarly $A_{ij}^{(2)}$ is obtained by integrating over Ω_2. Thus,

$$A_{ij} = A_{ij}^{(1)} + A_{ij}^{(2)}.$$

Let us construct a Schur complement system equivalent to system (4.18), by using block Gaussian elimination on system (4.18). Using row $r5$ in equation (4.18), we find that $u_{3,H} = 0$. Next, we solve for the rest of the unknowns in terms of $u_{4,H}$. Using rows $r1$ and $r2$ of equation (4.18), we obtain:

$$\begin{bmatrix} u_{1,H} \\ p_{1,H} \end{bmatrix} = - \begin{bmatrix} A_{11} & B_{11}^T \\ B_{11} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{14} \\ B_{14} \end{bmatrix} u_{4,H}.$$

Before we describe the equation satisfied by $u_{4,H}$, we describe how the mean value of the pressure, $p_{3,H}$, is determined in terms of the rest of the unknowns. Substituting the values obtained for $u_{1,H}, p_{1,H}, u_{3,H}$ and $u_{4,H}$ into row $r3$ of equation (4.18), we determine $p_{3,H}$ by

$$p_{3,H} = (-)\frac{A_{13}^T u_{1,H} + B_{13}^T p_{1,H} + A_{34}^{(1)} u_{4,H}}{B_{33}^T}.$$

To determine the equation satisfied by $u_{4,H}$, we substitute the values of $u_{1,H}, p_{1,H}$ and $u_{3,H}$, in terms of $u_{4,H}$, into row $r4$ of equation (4.18). We then obtain that

$$S_\Gamma^{(1)} u_{4,H} = G_4,$$

where

$$S_\Gamma^{(1)} \equiv A_{44}^{(1)} - \begin{bmatrix} A_{14} \\ B_{14} \end{bmatrix}^T \begin{bmatrix} A_{11} & B_{11}^T \\ B_{11} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{14} \\ B_{14} \end{bmatrix}.$$

Thus $S_\Gamma^{(1)}$ is the Schur complement associated with $u_{4,H}$ in system (4.18). $S_\Gamma^{(1)}$ is positive definite since $u_4^T S_\Gamma^{(1)} u_4$ equals the $a(.,.)$ norm of the *divergence free* discrete

velocity $[u_{1,H}, u_{3,H}, u_{4,H}]$ obtained by solving system (4.18) and that is positive definite; the proof is identical to that used to show that S_Γ is positive definite.

Similarly, using block Gaussian elimination on system (4.19), we obtain

$$S_\Gamma^{(2)} u_{4,H} = G_4,$$

where

$$S_\Gamma^{(2)} \equiv A_{44}^{(2)} - \begin{bmatrix} A_{24} \\ B_{24} \end{bmatrix}^T \begin{bmatrix} A_{22} & B_{22}^T \\ B_{22} & 0 \end{bmatrix}^{-1} \begin{bmatrix} A_{24} \\ B_{24} \end{bmatrix},$$

is the Schur complement associated with $u_{4,H}$ for the mixed boundary condition problem on subdomain Ω_2, namely problem (4.19). $S_\Gamma^{(2)}$ is also positive definite.

Since,

$$A_{44} = A_{44}^{(1)} + A_{44}^{(2)},$$

it follows that

$$S_\Gamma = S_\Gamma^{(1)} + S_\Gamma^{(2)}. \tag{4.20}$$

Thus, we could use either $S_\Gamma^{(1)}$ or $S_\Gamma^{(2)}$ as a preconditioner for the Schur complement S_Γ. Note that, solving linear systems involving $S_\Gamma^{(1)}$ or $S_\Gamma^{(2)}$, can be done without computing $S_\Gamma^{(1)}$ or $S_\Gamma^{(2)}$, by solving systems of the form (4.18) or (4.19), respectively.

We now consider the condition number of the preconditioned system. This determines the rate of convergence of the preconditioned conjugate gradient algorithm.

Theorem 7 *For $i = 1,2$, consider the preconditioner $S_\Gamma^{(i)}$ for the matrix S_Γ. There exists a positive constant $c_i(\Omega_1, \Omega_2)$ independent of h such that*

$$1 \le \frac{u_{4,H}^T S_\Gamma u_{4,H}}{u_{4,H}^T S_\Gamma^{(i)} u_{4,H}} \le 1 + c_i. \tag{4.21}$$

PROOF. To be specific, let $i = 1$. A similar argument holds for $i = 2$. Using equation (4.20), we see that

$$\frac{u_{4,H}^T S_\Gamma u_{4,H}}{u_{4,H}^T S_\Gamma^{(1)} u_{4,H}} = \frac{u_{4,H}^T S_\Gamma^{(1)} u_{4,H} + u_{4,H}^T S_\Gamma^{(2)} u_{4,H}}{u_{4,H}^T S_\Gamma^{(1)} u_{4,H}} \ge 1.$$

This proves the lower bound.

To obtain the upper bound, we consider

$$\frac{u_{4,H}^T S_\Gamma u_{4,H}}{u_{4,H}^T S_\Gamma^{(1)} u_{4,H}} = 1 + \frac{u_{4,H}^T S_\Gamma^{(2)} u_{4,H}}{u_{4,H}^T S_\Gamma^{(1)} u_{4,H}}.$$

Recall that $u_{4,H}^T S_\Gamma u_{4,H}$ is the $a(.,.)$ norm of the *divergence free* discrete velocity extension $[u_{1,H}, u_{2,H}, u_{3,H}, u_{4,H}]$, which is the velocity part of the solution of equation (4.11) with $-W_4$ defined by $-W_4 \equiv S_\Gamma u_{4,H}$. Similarly, $u_{4,H}^T S_\Gamma^{(1)} u_{4,H}$ is the $a(.,.)$ norm of the *divergence free* discrete velocity extension $[u_{1,H}, u_{3,H}, u_{4,H}]$, which is the velocity part of the solution of equation (4.18) with G_4 defined by $S_\Gamma^{(1)} u_{4,H}$. $u_{4,H}^T S_\Gamma^{(2)} u_{4,H}$ is the $a(.,.)$ norm of the *divergence free* discrete velocity extension $[u_{2,H}, u_{3,H}, u_{4,H}]$, which is the velocity part of the solution of equation (4.19) with G_4 defined by $S_\Gamma^{(2)} u_{4,H}$.

We seek to bound

$$u_{4,H}^T S_\Gamma^{(2)} u_{4,H} \leq c_1(\Omega_1, \Omega_2) u_{4,H}^T S_\Gamma^{(1)} u_{4,H},$$

for some positive constant $c_1(\Omega_1, \Omega_2)$. Since row $r1$ of equation (4.19) is homogeneous, it follows that $[u_{2,H}, u_{3,H}, u_{4,H}]$ is $a(.,.)$-orthogonal to all *divergence free* discrete velocities in $X_h(\Omega_2) = V_h(\Omega_2) \cap \vec{H}_{0,\partial\Omega_2}(div, \Omega_2)$. Thus, $u_{4,H}^T S_\Gamma^{(2)} u_{4,H}$ is less than or equal to the $a(.,.)$ norm of any *discrete divergence free* function in $V_h(\Omega_2)$ with the same *normal trace* as $[u_{2,H}, u_{3,H}, u_{4,H}]$. Note that, the *normal trace* of $[u_{2,H}, u_{3,H}, u_{4,H}]$ is zero on $\Gamma_2^c = \partial\Omega_2 - \Gamma$, and equals $u_{4,H}$ on Γ (up to sign). Thus, in particular, $u_{4,H}^T S_\Gamma^{(2)} u_{4,H}$ is less than or equal to the $a(.,.)$ norm of the *discrete divergence free* extension $E^h(\gamma_n[u_{2,H}, u_{3,H}, u_{4,H}])|_{\partial\Omega_2}$, given by theorem 2 (the Extension theorem), in Chapter 1. By the Extension theorem, the $a(.,.)$ norm of the extension is bounded by the $H^{-1/2}(\partial\Omega_2)$ norm of the *normal trace*, with a constant independent of h.

Next, we note that the $H^{-1/2}(\partial\Omega_2)$ norm of $\gamma_n[u_{2,H}, u_{3,H}, u_{4,H}]|_{\partial\Omega_2}$ can be bounded by the $H^{-1/2}(\partial\Omega_1)$ norm of $\gamma_n[u_{1,H}, u_{2,H}, u_{3,H}, u_{4,H}]|_{\partial\Omega_1}$, with a constant independent of h, but possibly depending on the two subdomains. This is because, we can map $\partial\Omega_1$ onto $\partial\Omega_2$ by a bijective mapping, keeping Γ fixed. This establishes an equivalence between $H^{1/2}(\partial\Omega_1)$ and $H^{1/2}(\partial\Omega_2)$. By using duality, we obtain the result. Next, we note that we can bound the $H^{-1/2}(\partial\Omega_1)$ norm of $\gamma_n[u_{1,H}, u_{2,H}, u_{3,H}, u_{4,H}]|_{\partial\Omega_1}$ by the $a(.,.)$ norm of $[u_{1,H}, u_{3,H}, u_{4,H}]$, with a constant independent of h, by using Lemma 4 (the *normal trace* Lemma). But the square of this norm is equal to $u_{4,H}^T S_\Gamma^{(1)} u_{4,H}$. Thus, there exists a positive constant $c_i(\Omega_1, \Omega_2)$, independent of h, such that the upper bound in equation (4.21) holds. □

REMARK. Note that $S_\Gamma^{(i)}$, maps the Neumann data on Γ onto Dirichlet data on Γ. This is done by finding the Dirichlet data of a discrete velocity satisfying the piecewise homogeneous equation in (4.18) and satisfying the given Neumann boundary

condition on Γ. Thus, we may refer to $S_\Gamma^{(i)}$ as a Neumann-Dirichlet map. Since

$$S_\Gamma^{(i)} : (H_{00}^{1/2}(\Gamma))' \longrightarrow H_{00}^{1/2}(\Gamma),$$

we may expect heuristically that its condition number be $O(\frac{1}{h})$, since the difference in exponents of the spaces is one, corresponding to a derivitive. Since S_Γ is spectrally equivalent to $S_\Gamma^{(i)}$, we also expect S_Γ to have a condition number of $O(\frac{1}{h})$. Since $S_\Gamma = S_\Gamma^{(1)} + S_\Gamma^{(2)}$, we see that S_Γ maps the Neumann data on Γ to the jump in Dirichlet data on Γ, obtained by using the two Neumann-Dirichlet maps on the two subdomains. Thus, system (4.12) has as its solution, the appropriate Neumann data that leads to the correct jump in the Dirichlet data. For details about the terminology of the *Dirichlet-Neumann* algorithm, see Bjørstad and Widlund [4].

REMARK ON QUANTITATIVE BOUNDS. We note that we can easily obtain some quantitative bounds for the condition number of the *Dirichlet-Neumann* algorithm in the case of special geometries and special coefficients for the elliptic operator. Suppose that Ω_2 is the mirror image of Ω_1, under reflection across Γ. Further, assume that the mesh on Ω_2 also is a reflection of the mesh on Ω_1, and that the coefficients $\mathcal{A}(x)$ are even across Γ. Then, we can bound $u_{4,H}^T S_\Gamma^{(2)} u_{4,H}$ by $u_{4,H}^T S_\Gamma^{(1)} u_{4,H}$ since the even extension of $[u_{1,H}, u_{3,H}, u_{4,H}]$ across Γ into Ω_2 provides a *divergence free* harmonic extension as used in the proof of Theorem 7. This shows the condition number to be bounded by 2. The same bound holds if the mirror image across Γ, of the mesh on Ω_1 is contained in the mesh on Ω_2.

Bibliography

[1] G. P. Astrakhantsev, *Method of fictitious domains for a second order elliptic equation with natural boundary conditions*, U.S.S.R. Computational Math. and Math. Phys. 18 (1978), pp. 114-121.

[2] O. A. Axelsson and V. A. Barker, *Finite element solution of boundary value problems*, Academic Press, 1984.

[3] I. Babuška and A. K. Aziz, *Survey lectures on the mathematical foundations of the finite element method* in The Mathematical Foundation of the Finite Element Method with Applications to Partial Differential Equations. A. K. Aziz and I. Babuška, editors. Academic Press, New York, 1972.

[4] P. E. Bjørstad and O. B. Widlund, *Iterative methods for the solution of elliptic problems on regions partitioned into substructures,* SIAM J. Numer. Anal. 23, 1986, 1097-1120.

[5] J. H. Bramble, J. E. Pasciak and A. H. Schatz, *An iterative method for elliptic problems on regions partitioned into substructures,* Math. Comp. 46, 1986, 361-369.

[6] J. H. Bramble, R. E. Ewing, J. E. Pasciak and A. H. Schatz, *A preconditioning technique for the efficient solution of problems with local grid refinement*, pages 149-159, Computer Methods in Applied Mechanics and Engineering, Vol. 67, 1988.

[7] J. H. Bramble, J. E. Pasciak and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, I*, Math. Comp. Vol. 47, No. 175, pages 103-134, 1986.

[8] J. H. Bramble, J. E. Pasciak and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, II*, Math. Comp. Vol. 49, pages 1-16, 1987.

[9] J. H. Bramble, J. E. Pasciak and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, III*, Cornell University, 1987.

[10] J. H. Bramble, J. E. Pasciak and A. H. Schatz, *The construction of preconditioners for elliptic problems by substructuring, IV*, Cornell University, 1988. To appear in Math. Comp.

[11] X. C. Cai, *Some domain decomposition algorithms for non-selfadjoint elliptic and parabolic partial differential equations,* Doctoral Thesis, New York University, New York, 1989.

[12] P. Ciarlet, *Numerical analysis of the finite element method*, University of Montreal Press, 1976.

[13] M. Dryja and O. B. Widlund, *Some domain decomposition algorithms for elliptic problems,* Tech. Rep. 438, also Ultracomputer Note 155, Department of Computer Science, Courant Institute, April 1989. To appear in the proceeding of the Conference on Iterative Methods for Large Linear Systems held in Austin, Texas, October 1988, to celebrate the Sixty-fifth Birthday of David M. Young, Jr.

[14] M. Dryja and Olof B. Widlund, *On the Optimality of an Additive Iterative Refinement Method,* Department of Computer Science, Courant Institute, Tech. Rep. 442, also Ultracomputer Note 156, To appear in the proceeding of the Fourth Copper Mountain Conference on Multigrid Methods, held at Copper Mountain, Colorado, April 9 - 14, 1989.

[15] V.Girault and P.A. Raviart, *Finite element methods for Navier-Stokes equations*, Springer-Verlag, 1986.

[16] R. Glowinski and M. F. Wheeler, *Domain decomposition and mixed finite element methods for elliptic problems,* Proceedings of First International Symposium on

Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1988.

[17] Gene H. Golub and Charles F. Van Loan, *Matrix computations,* Johns Hopkins Univ. Press, 1983.

[18] Anne Greenbaum, Congming Li, and Han Zheng Chao, *Parallelizing preconditioned conjugate gradient algorithms,* Technical report, Courant Institute, 1988. To appear in Computer Physics Communications.

[19] P. Grisvard, *Elliptic problems in nonsmooth domains,* Pitman, Boston, 1985.

[20] L. A. Hageman and D. M. Young, *Applied iterative methods,* Academic Press, 1981.

[21] J. L. Lions and E. Magenes, *Non-homogeneous boundary value problems and applicatons,* Volume I, Springer-Verlag, 1972.

[22] P. L. Lions, *On the Schwarz alternating method. I.*, in First International Symposium on Domain Decomposition Methods for Partial Differential Equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1988.

[23] J. F. Maitre and F. Musy, *The contraction number of a class of two level methods; An exact evaluation for some finite element subspaces and model problems,* in Multigrid Methods, Proceedings, W. Hackbusch and U. Trottenberg, eds., Lecture Notes in Mathematics 960, Springer-Verlag, Berlin, 1982.

[24] J. Mandel and S. McCormick, *Iterative solution of elliptic equations with refinement: The two-level case,* in Second International Symposium on Domain Decomposition Methods for Partial Differential Equations, T. Chan, R. Glowinski, G. A. Meurant, J. Périaux, and O. Widlund, eds., SIAM, Philadelphia, 1989.

[25] A. M. Matsokin, *Norm-preserving prolongations of mesh functions*, Soviet Journal of Numerical Analysis and Mathematical Modelling, Vol. 3, 137-149, 1988.

[26] S. McCormick and J. Thomas, *The fast adaptive composite grid (FAC) method for elliptic equations,* Math. Comp. 46, 1986, 439-456.

[27] J.Nečas, *Les méthodes directes en théorie des equations elliptiques,* Masson, 1967.

[28] J. E. Pasciak, *Two domain decomposition techniques for Stokes problems,* in Domain Decomposition Methods, SIAM, Tony Chan, Roland Glowinski, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, Philadelphia, 1989.

[29] A. Quarteroni, *Domain decomposition algorithms for Stokes equations,* in Domain Decomposition Methods, SIAM, Tony Chan, Roland Glowinski, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, Philadelphia, 1989.

[30] P. A. Raviart and J. M. Thomas, *A mixed finite element method for second order elliptic problems,* in Mathematical Aspects of the Finite Element Method, Lecture Notes in Mathematics, Springer-Verlag, Heidelberg, 1977.

[31] G. Strang and G. Fix, *An analysis of the finite element method,* Prentice-Hall,1973.

[32] J. M. Thomas, *Sur l'analyse numerique des methodes d'elements finis hybrides et mixtes,* Doctoral Thesis, Université Pierre et Marie Curie, Paris, 1977.

[33] M. F. Wheeler and R. Gonzalez, *Mixed finite element methods for petroleum reservoir engineering problems,* in Computing Methods in Applied Sciences and Engineering, Volume VI. R. Glowinski and J. L. Lions (Editors), North Holland, 1984.

[34] Olof B. Widlund, *An extension theorem for finite element spaces with three applications,* in Numerical Techniques in Continuum Mechanics, pages 110-122, Notes on Numerical Fluid Mechanics, v. 16, Friedr. Vieweg und Sohn, Braunschweig/Wiesbaden, 1987. Wolfgang Hackbusch and Kristian Witsch, editors. Proceedings of the Second GAMM Seminar, Kiel, January, 1986.

[35] Olof B. Widlund, *On the rate of convergence of the classical Schwarz alternating method in the case of more than two subregions,* Technical report, Department of Computer Science, Courant Institute, 1988.

[36] Olof B. Widlund, *Optimal iterative refinement methods,* in Domain Decomposition Methods, SIAM, Tony Chan, Roland Glowinski, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, Philadelphia, 1989.

[37] Olof B. Widlund, *Iterative substructuring methods: Algorithms and theory for elliptic problems in the plane,* in Roland Glowinski, Gene H. Golub, Gérard A. Meurant, Jacques Périaux, and Olof Widlund, editors, Domain Decomposition Methods for Partial Differential Equation, SIAM, Philadelphia, 1988.

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iterative refinement
methods for mixed... c.1